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D-Branes And Mirror Symmetry

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Abstract

We study $(2, 2)$ supersymmetric field theories on two-dimensional worldsheet with boundaries. We determine D-branes (boundary conditions and boundary interactions) that preserve half of the bulk supercharges in non-linear sigma models, gauged linear sigma models, and Landau-Ginzburg models. We identify a mechanism for brane creation in LG theories and provide a new derivation of a link between soliton numbers of the massive theories and R-charges of vacua at the UV fixed point. Moreover we identify Lagrangian submanifolds that arise as the mirror of certain D-branes wrapped around holomorphic cycles of Kähler manifolds. In the case of Fano varieties this leads to the explanation of Helix structure of the collection of exceptional bundles and soliton numbers, through Picard-Lefshetz theory applied to the mirror LG theory. Furthermore using the LG realization of minimal models we find a purely geometric realization of Verlinde Algebra for $SU(2)$ level k as intersection numbers of D-branes. This also leads to a direct computation of modular transformation matrix and provides a geometric interpretation for its role in diagonalizing the Fusion algebra.

Contents

1 Introduction

Two-dimensional quantum field theories formulated on a surface with boundaries are important arena in various fields of study. Among others they provide the starting point for open string theories, in particular for the study of D-branes. D-branes have proven to be indispensable elements in string theories. The interplay between the target space properties of D-branes, as sources for RR fields [1], and how they couple to string worldsheet has been very important. In most of the recent studies of D-branes from the worldsheet point of view, boundary conformal field theories, namely conformal field theories with conformally invariant boundary conditions, have been the focus of attention. However, we believe that one can learn more, especially on the off-shell properties of string theories, by studying quantum field theories on a worldsheet with boundaries where the bulk theories and/or the boundary conditions are not necessarily conformally invariant. We also expect that such a study may open a way to reveal the geometric principle of string theories.

One of the main aims of this work is to initiate the study of D-brane associated with quantum field theories with and without conformal invariance, employing supersymmetry as the basic constraint. In particular we study D-branes in supersymmetric sigma models on Kähler manifolds and their mirror description [2] in terms of D-branes in Landau-Ginzburg theories. We will consider both the conformal case (where target space has

vanishing c_1 and is a CY manifold) as well as asymptotically free theories (where $c_1 \geq 0$) which have mass gaps in many cases. We will mainly consider D-branes corresponding to holomorphic cycles on the Kähler manifold which are mirror, as we will discuss, to Lagrangian submanifolds on the Landau-Ginzburg side.

Along the way we find some interesting similarities and differences between various aspects of D-branes for the massive sigma models and the conformal one. In particular we see how Brane creation also occurs for massive theories as we change the parameters in the sigma model. We also define a notion of intersection between two Lagrangian D-branes in the massive theory which is a refined version of the classical intersection of the cycles in the Calabi-Yau realization of it (in particular the inner product in the massive case is neither symmetric nor anti-symmetric). Also we apply the machinery of D-branes that we develop for Landau-Ginzburg theories to the LG realization of $\mathcal{N} = 2$ minimal models. In this way we find a purely geometric realization of Verlinde algebra for bosonic $SU(2)$ WZW model at level k, as well as the modular transformation matrix. Also we are able to shed light on an old observation of Kontsevich connecting "helices of exceptional bundles" on Fano varieties with soliton numbers of certain Landau-Ginzburg theories, as a consequence of mirror symmetry acting on D-branes.

The organization of this paper is as follow. In section 2 we review aspects of LG solitons in $\mathcal{N} = 2$ theories [3, 4]. In section 3 we discuss D-branes for supersymmetric sigma models and LG theories. In this section we will consider both holomorphic and Lagrangian D-branes. For the most of this paper we will mainly concentrate on holomorphic D-branes ("B-type") in the context of supersymmetric sigma model and Lagrangian D-branes ("A-type") in the context of LG models. We define and study boundary states corresponding to such D-branes following the study in string theory and conformal field theory [5–7]. In section 4 we discuss the phenomenon of D-brane creation of massive LG theories, and show how these results give a reinterpretation of the connection between R-charges of chiral fields at the ultra-violet fixed point and the soliton numbers of its massive deformation discovered in [3] . In section 5 we apply these results to the study of $\mathcal{N} = 2$ minimal models and show how aspects of conformal theory, including certain properties of Cardy states, and its relations with Verlinde algebra, as well as its overlap with Ishibashi states in terms of modular transformation matrix, can be derived in a purely geometric way. In section 6 we derive, using the results of [2] the mirror of certain D-branes on Fano varieties in terms of D-branes in the mirror LG models. In section 7 we apply the study of D-branes to the LG mirrors of Fano varieties and uncover beautiful mirror interpretation for helices of exceptional bundles on Fano varieties in terms of D-branes of the LG mirror. In section 8 we discuss connecting the LG mirror for the case of non-compact geometries

in Calabi-Yau (such as $\mathbb{P}^2$ inside a CY threefold) discussed in [2] to a local non-compact geometric mirror as was used in [8, 9]. Moreover we show how this is related in the case of local non-compact threefolds to the probe description in F-theory and its BPS states.

While completing this work, a paper [10], which has some overlap with our discussions in sections 3 and 5, appeared.

2 BPS Solitons in $\mathcal{N}$=2 Landau Ginzburg Theories

In this section, we review some basic facts on Landau-Ginzburg models, especially on the spectrum of BPS solitons and the relation to Picard-Lefshetz theory of vanishing cycles. The action for a Landau Ginzburg model of n chiral superfields Φ_i $(i = 1, \dots, n)$ with superpotential $W(\Phi)$ is given by

$$S = \int \mathrm{d}^2x \left[\int \mathrm{d}^4\theta\, K(\Phi_i, \overline{\Phi}_i) + \tfrac{1}{2} \left(\textstyle\int \mathrm{d}^2\theta\, W(\Phi_i) + \int \mathrm{d}^2\bar{\theta}\, \overline{W}(\overline{\Phi}_i) \right) \right] . \tag{2.1}$$

Here $K(\Phi_i, \bar{\Phi}_i)$ is the Kähler potential which defines the Kähler metric $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K(\Phi_i, \bar{\Phi}_i)$. If the superpotential $W(\Phi)$ is a quasi-homogeneous function with an isolated critical point (which means $dW = 0$ can only occur at $\Phi_i = 0$) then the above action for a particular choice of $K(\Phi, \bar{\Phi})$ is believed to define a superconformal theory [11, 12]. For a general superpotential the vacua are labeled by critical points of W, i.e., where

$$\phi^i(x) = \phi^i_*\,, \quad \partial_i W|_{\phi^i_*} = 0\,. \tag{2.2}$$

The theory is purely massive if all the critical points are isolated and non-degenerate, which means that near the critical points W is quadratic in fields. We assume this and label the non-degenerate critical points as $\{\phi_a \,|\, a = 1, \cdots, N\}$. In such a case the number of vacua of the theory is equal to the dimension of the local ring of $W(\Phi)$, $\mathcal{R} = \frac{\mathbb{C}[\Phi]}{\partial_{\phi^i} W}$. When we have more than one vacuum we can have solitonic states in which the boundary conditions of the fields at the left spatial infinity $x^1 = -\infty$ is at one vacuum and is different from the one at right infinity $x^1 = +\infty$ which is in another vacuum. The geometry of solitons and their degeneracies have been extensively studied in [3, 4] which we will now review.

Consider a massive Landau Ginzburg theory with superpotential $W(\Phi_i)$. Solitons are static solutions, $\phi^i(x^1)$, of the equations of motion interpolating between *different* vacua i.e., $\phi^i(-\infty) = \phi^i_a$ and $\phi^i(+\infty) = \phi^i_b$, $a \neq b$. The energy of a static field configuration interpolating between two vacua is given by [13]

$$E_{ab} \;=\; \int_{-\infty}^{+\infty} dx^1 \left\{ g_{i\bar{j}} \frac{d\phi^i}{dx^1} \frac{d\bar{\phi}^{\bar{i}}}{dx^1} + \tfrac{1}{4} g^{i\bar{j}} \partial_i W \partial_{\bar{j}} \bar{W} \right\} \tag{2.3}$$

$$= \int_{-\infty}^{+\infty} dx^1 \left| \frac{d\phi^i}{dx^1} - \tfrac{\alpha}{2} g^{i\bar{j}} \partial_{\bar{j}} \bar{W} \right|^2 + \mathrm{Re}\Big((\bar{\alpha}(W(b) - W(a)) \Big) .$$

Where $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$ is the Kähler metric and α is an arbitrary phase. By choosing an appropriate α we can maximize the second term. Since α is a phase it is clear that the second term is maximum when phase of $W(b) - W(a)$ is equal to α. This implies a lower bound on the energy of the configuration,

$$E_{ab} \geq |W(b) - W(a)|. \tag{2.4}$$

In fact the central charge in the supersymmetry algebra in this sector is $(W(b) - W(a))$. BPS solitons saturate this bound and therefore satisfy the equation,

$$\frac{d\phi^i}{dx^1} = \frac{\alpha}{2} g^{i\bar{j}} \partial_{\bar{j}} \bar{W} \,, \quad \alpha = \frac{W(b) - W(a)}{|W(b) - W(a)|} \,. \tag{2.5}$$

An important consequence of the above equation of motion of a BPS soliton is that along the trajectory of the soliton the superpotential satisfies the equation,

$$\partial_{x^1} W = \frac{\alpha}{2} g^{i\bar{j}} \partial_i W \partial_{\bar{j}} \bar{W} \,. \tag{2.6}$$

Now since the metric $g^{i\bar{j}}$ is positive definite, we know $g^{i\bar{j}} \partial_i W \partial_{\bar{j}} \bar{W}$ is real, and therefore the image of the BPS soliton in the W-plane is a straight line connecting the corresponding critical values $W(a)$ and $W(b)$.

The number of solitons between two vacua is equal to the number of solutions of eq. (2.5) satisfying the appropriate boundary conditions. The general way to count the number of solitons has been determined in [3] and we will review it in the next subsection. Here we note that for the case of a single chiral superfield the number of solitons between two vacua can also be determined using eq. (2.6). Since the image of the soliton trajectory is a straight line in the W-plane therefore by looking at the pre-image of the straight line connecting the corresponding critical values in the W-plane we can determine the number of solitons between the two vacua. But since the map to the W-plane is many to one, not every pre-image of a straight line in the W-plane is a soliton. It is possible for the trajectory to start at a critical point follow a path whose image is a straight line in the W-plane and end on a point which is not a critical point but whose image in the W-plane is a critical value. The BPS solitons are those pre-images of the straight line in the W-plane which start and end on the critical points.

2.1 Vanishing cycles

It was shown in [3] that the soliton numbers also have a topological description in terms of intersection numbers of vanishing cycles. The basic idea is to solve the soliton

equation (2.5) along all possible directions emanating from one of the critical points. In other words the idea is to study the "wave front" of all possible solutions to the (2.5).

With no loss of generality we may assume $\alpha = 1$. Near a critical point ϕ_a^i we can choose coordinates u_a^i such that,

$$W(\phi) = W(\phi_a) + \sum_{i=1}^{n} (u_a^i)^2 \,. \tag{2.7}$$

In this case it is easy to see that the solutions to (2.5) will have an image in the W-plane which is on a positive real line starting from $W(\phi_a)$. Consider a point w on this line. Then the space of solutions to (2.5) emanating from $u_a^i = 0$ over this w is a real $(n-1)$ dimensional sphere defined by

$$\sum_{i=1}^{n} (\mathrm{Re}(u_a^i))^2 = w - w_a \,, \quad \mathrm{Im}(u_a^i) = 0 \,. \tag{2.8}$$

where $w_a = W(\phi_a)$. Note that as we take $w \mapsto w_a$ the sphere vanishes. This is the reason for calling these spheres "vanishing cycles". As we move away, the wavefront will no longer be as simple as near the critical point, but nevertheless over each point w on the positive real line emanating from $w_a = W(\phi_a)$ the pre-image is a real $(n-1)$ dimensional homology cycle Δ_a in the $n-1$ complex manifold defined by $W^{-1}(w)$. Similarly as we move from w_b toward w_a there is a cycle Δ_b evolving according to the soliton equation eq. (2.5) (this would correspond to $\alpha = -1$). For a fixed value of w we can compare Δ_a and Δ_b. Solitons originating from ϕ_a and traveling all the way to ϕ_b correspond to the points in the intersection $\Delta_a \cap \Delta_b$. This number, counted with appropriate signs is the intersection number of the cycles, $\Delta_a \circ \Delta_b$. The intersection number counts the number of solitons weighed with $(-1)^F$ for the lowest component of each soliton multiplet [3]. This is independent of deformation of the D-terms. In particular this measures the net number of solitons that cannot disappear by deformations in the D-terms. We will denote this number by A_{ab} and sometimes loosely refer to it as the number of solitons between a and b. We thus have

$$A_{ab} = \Delta_a \circ \Delta_b. \tag{2.9}$$

Note that to calculate the intersection numbers we have to consider the two cycles Δ_a and Δ_b in the same manifold $W^{-1}(w)$. Since the intersection number is topological, a continuous deformation does not change them and hence we can actually calculate them using some deformed path in the W-plane (rather than the straight line) as long as the path we are choosing is homotopic to the straight line. How we transport the cycle along the path will not change the intersection numbers as that is topological and nothing is

discontinuous, as long as the paths have the same homotopy class in the W-plane with the critical values deleted. One way, but not the only way, to transport vanishing cycles along arbitrary paths, is to use the soliton equation (2.5) but instead of having a fixed α, as would be the case for a straight line, choose α to be $e^{i\theta}$ where θ denotes the varying slope of the path.

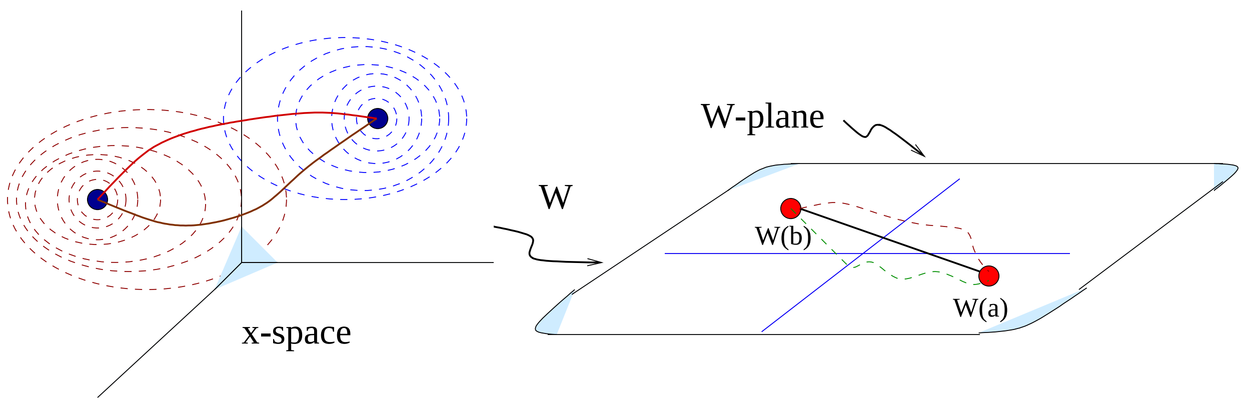

Figure 1: BPS soliton map to straight line in the W-plane. Soliton solutions exist for each intersection point of vanishing cycles. Lines in the W-plane which are homotopic to the straight line (doted lines) can also be used to calculate soliton numbers.

Let us fix a point w in the W-plane. For each critical point a of W, we choose an arbitrary path in the W-plane emanating from $W(a)$ and ending on w, but not passing through other critical values. This yields N cycles Δ_a over $W^{-1}(w)$ and it is known [14] that these form a complete basis for the middle-dimensional homology cycles of $W^{-1}(w)$. Moreover, if we choose different paths the vanishing cycle we get is a linear combination of the above and the relation between them is known through the Picard Lefshetz theory as we will now review.

2.2 Picard-Lefshetz monodromy

As we have discussed the basis for the vanishing cycle over each point w in the W-plane depends on the choice of paths connecting it to the critical point. Picard-Lefshetz monodromy relates how the basis changes if we change paths connecting w to the critical values. This is quite important for the study of solitons and leads to a jump in the soliton numbers. To explain the physical motivation for the question, consider three critical values $W(a), W(b)$ and $W(c)$ depicted in Fig. 2(a), with no other critical values nearby. Suppose we wish to compute the number of solitons between them. According to our

discussion above we need to connect the critical values by straight lines in the W-plane and ask about the intersection numbers of the corresponding cycles. As discussed above this is the same, because of invariance of intersection numbers under deformation, as the intersection numbers of the vanishing cycles over the point w connecting to the three critical values as shown in Fig. 2(a). Thus the soliton number is $A_{ij} = \Delta_i \circ \Delta_j$. However suppose now that we change the superpotential W so that the critical values change according to what is depicted in Fig. 2(b), and that the $W(b)$ passes through the straight line connecting $W(a)$ and $W(c)$. In this case to find the soliton numbers between the a vacuum and the c vacuum we have to change the homotopy class of the path connecting w to the critical value $W(a)$ as depicted by Fig. 2(b).

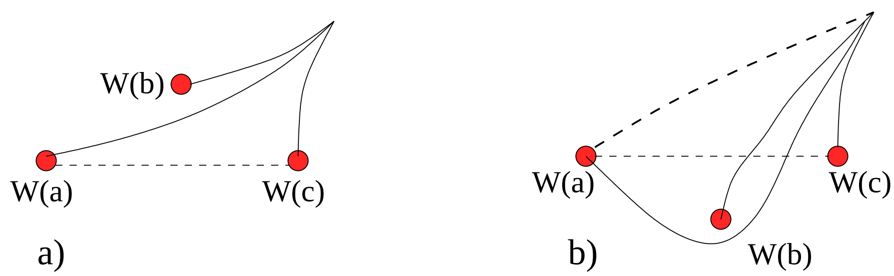

Figure 2: As the positions of critical values change in the W-plane, the choice of the vanishing cycles relevant for computing the soliton numbers change.

In particular the homology element corresponding to vanishing cycle a changes $\Delta_a \to \Delta_a'$ and we need to find out how it changes. Picard-Lefshetz theory gives a simple formula for this change. In particular it states that

$$\Delta_a' = \Delta_a \pm (\Delta_a \circ \Delta_b)\Delta_b \, . \tag{2.10}$$

The sign in the above formula is determined once the orientation of the cycles are fixed and will depend on the handedness of the crossing geometry (see [3]). This is perhaps most familiar to string theorists in the context of moduli space of Riemann surfaces, where if we consider a point on the moduli space of Riemann surfaces where a 1-cycle shrinks to zero, as we go around this point, all the other cycles intersecting it will pick up a monodromy in the class of the vanishing cycle (the case of the torus and the $\tau \to \tau + 1$ is the most familiar case, where the b cycle undergoes a monodromy $b \to b + a$).

As a consequence of the above formula we can now find how the number of solitons between the a and the c vacuum change. We simply have to take the inner product $\Delta_a' \circ \Delta_c$ and we find

$$A_{ac}' = A_{ac} \pm A_{ab} A_{bc} \, .$$

2.3 Non-compact n Cycles

An equivalent description which will be important for later discussion involves defining soliton numbers in terms of the intersection numbers of n real dimensional non-compact cycles which are closely related to the $n-1$ dimensional vanishing cycles we have discussed. The idea is to consider the basis for the vanishing cycles in the limit where the point $w \to e^{i\theta}\infty$. Let us consider the case where $\theta = 0$. In this case we are taking w to go to infinity along the positive real axis. Let us assume that the imaginary part of the critical values are all distinct. In this case a canonical choice of paths to connect the critical points to w is along straight lines starting from the critical values $W(a)$ stretched along the positive real axis. We denote the corresponding non-compact n dimensional cycles by γ_a. Then we have

$$W(\gamma_a) = I_a\,, \qquad \text{and} \quad \partial\gamma_a \cong \Delta_a\Big|_{w\mapsto +\infty}\,, \tag{2.11}$$

where

$$I_a \equiv \{w_a + t \mid t \in [0,\infty)\}\,. \tag{2.12}$$

Two such cycles are shown in Figure 3.

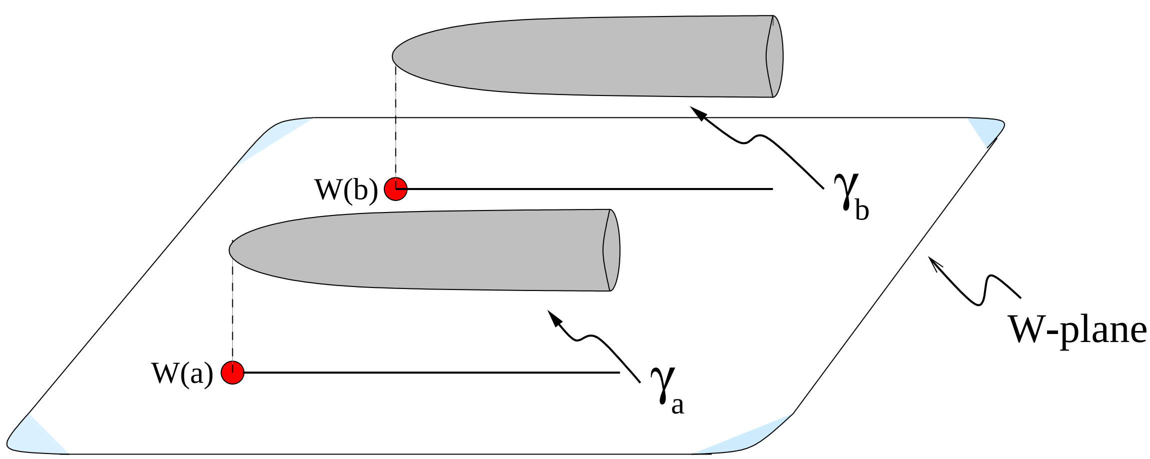

Figure 3: The cycles emanating from the critical points. The images in the W-plane are the straight lines emanating from the critical values and extending to the infinity in the real positive direction.

Let B be the region of $\mathbf{C}^n$ where ReW is larger than a fixed value which is chosen sufficiently large. The non-compact cycles γ_a can be viewed as elements of the homology

group $H_n(\mathbf{C}^n, B)$ corresponding to n-cycles with boundary in B, and again it can be shown [14] that they provide a complete basis for such cycles.

For a pair of distinct critical points, a and b, the non-compact cycles γ_a and γ_b do not intersect with each other, since their images in the W-plane are parallel to each other (and are separate from each other in the present situation). In this situation we consider deforming the second cycle γ_b so that its image in the W-plane is rotated with an infinitesimally small positive angle ϵ against the real axis. We denote this deformed cycle by γ'_b. We define the "intersection number" of γ_a and γ_b as the geometric intersection number of γ_a and γ'_b. Depending on whether $\mathrm{Im}W(a)$ is smaller or larger than $\mathrm{Im}W(b)$, the images of γ_a and γ'_b in the W-plane do not or do intersect with each other. In the

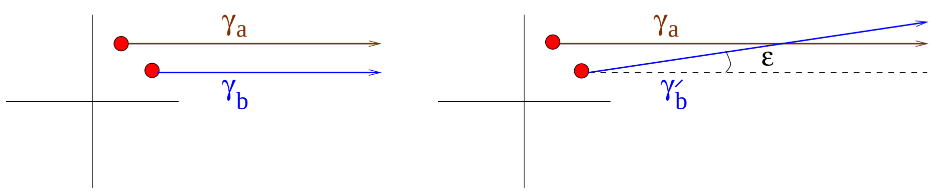

Figure 4: The images in the W-plane of γ_a and γ_b (left); and γ_a and γ'_b (right). The second will give rise to"intersection number". As we will see in the next section, this contains a certain information on D-branes in the LG model.

former case the "intersection number" is of course zero. In the latter case as shown in Fig. 4, the intersection number $\gamma_a \circ \gamma'_b$ is counted by going to the point on the W-plane where their images intersect and asking what is the intersection of the corresponding vanishing cycles $\Delta_a \circ \Delta_b$. Thus the intersection of these n-dimensional cycles has the information about the soliton numbers. In particular if there are no extra critical values between the I_a and I_b we will have

$$\gamma_a \circ \gamma'_b = A_{ab}, \quad a \neq b\,. \tag{2.13}$$

If there are extra critical values between I_a and I_b then these intersection numbers are related to the soliton numbers by the Picard-Lefshetz action as discussed before.

In the next section we will see that the cycles γ_a defined through parallel transport by the soliton equation (2.5) can be viewed as D-branes for LG models that preserve half of the supersymmetries on the worldsheet. There we will also see that the "intersection number" of γ_a and γ_b as defined above can be interpreted as the supersymmetric index for the worldsheet theory of open strings stretched between these cycles.

2.4 Examples

In this section we are going to discuss some examples for the soliton numbers in the case of LG models. We will concentrate on LG models representing $\mathcal{N} = 2$ minimal models as well as the LG models mirror to $\mathbb{P}^N$ sigma models.

2.4.1 Deformed $\mathcal{N} = 2$ Minimal models

$\mathcal{N} = 2$ minimal models are realized as the infra-red fixed point of LG models [11, 12] . The soliton numbers for the deformed version of these theories has been studied in detail for the massive deformations of the A-series minimal models [13, 3], which we will now review.

The k-th minimal model is described by an LG theory with one chiral superfield X with superpotential

$$W(X) = \tfrac{1}{k+2} X^{k+2} \,. \tag{2.14}$$

If we add generic relevant operators to the superpotential we can deform this theory to a purely massive theory. In this case we will get $k+1$ vacua and we can ask how many solitons we get between each pair. For example if we consider the integrable deformation,

$$W(X) = \tfrac{1}{k+2} X^{k+2} - X, \tag{2.15}$$

then there are $k+1$ vacua which are solutions of $dW = 0$ given by $X = e^{\frac{2\pi i n}{k+1}}$, $n = 0, \cdots, k$. In this case one can count [13] the preimage of the straight lines in the W-plane and ask which ones connect critical points and in this way compute the number of solitons. It turns out that in this case there is exactly one soliton connecting each pair of critical points. If we deform W the number of solitons will in general change as reviewed above. In this case one can show (by taking proper care of the relevant signs in the soliton number jump) that there is always at most one soliton between vacua. The precise number can be determined starting from the above symmetric configuration (see [3]). The analog of the non-compact 1-cycles γ_i in this case will be discussed in more detail in section 4 after we discuss their relevance as D-branes in section 3. They are cycles in the X-plane which asymptote an $(k+2)$-th root of unity as $X \to \infty$. That there are $k+1$ inequivalent such homology classes for $H_1(\mathbf{C}, ReW = \infty)$ is related to the fact that there are $k+1$ such classes defined by γ's up to linear combinations.

2.4.2 $\mathbb{P}^{N-1}$

We next consider the $\mathbb{P}^{N-1}$ sigma model. The soliton matrix of the non-linear sigma model with target space $\mathbb{P}^{N-1}$ can be computed directly by studying the tt^* equations [3] and their relations to soliton numbers [4]. This has been done in [15, 16]. The tt^* equations are, however, very difficult to solve for more non-trivial spaces such as toric del Pezzos. The mirror LG theory obtained in [2] provides a simple way of calculating the soliton matrix. We start with the case $N = 2$ where we can present explicit solutions to the soliton equation.

The Landau Ginzburg theory which is mirror to the non-linear sigma model with $\mathbb{P}^1$ target space is the $\mathcal{N} = 2$ sine-Gordon model with the superpotential,

$$W(x) = x + \frac{\lambda}{x}\,. \tag{2.16}$$

Here $x = \mathrm{e}^{-y}$ is the single valued coordinate of the cylinder $\mathbb{C}^{\times}$ and $-\log\lambda$ corresponds to the Kähler parameter of $\mathbb{P}^1$. The critical points are $x_*^{\pm} = \pm\sqrt{\lambda}$ with critical values $w_*^{\pm} = \pm 2\sqrt{\lambda}$. As mentioned in the previous section the BPS solitons are trajectories, $x(t)$, starting and ending on the critical points such that their image in the W-plane is a straight line,

$$x(t) + \frac{\lambda}{x(t)} = 2\sqrt{\lambda}(2t-1)\,, \;\; t \in [0,1]\,. \tag{2.17}$$

This is a quadratic equation with two solutions given by,

$$x(t)_{\pm} = \sqrt{\lambda}(2t-1) \pm 2i\sqrt{\lambda}\sqrt{t-t^2} = \sqrt{\lambda}e^{\pm i\tan^{-1}\frac{2\sqrt{t-t^2}}{2t-1}}\,. \tag{2.18}$$

Since $x_+(t) = x_-(t)^*$ and $|x_+(t)| = |\sqrt{\lambda}|$, there are two solitons between the two vacua

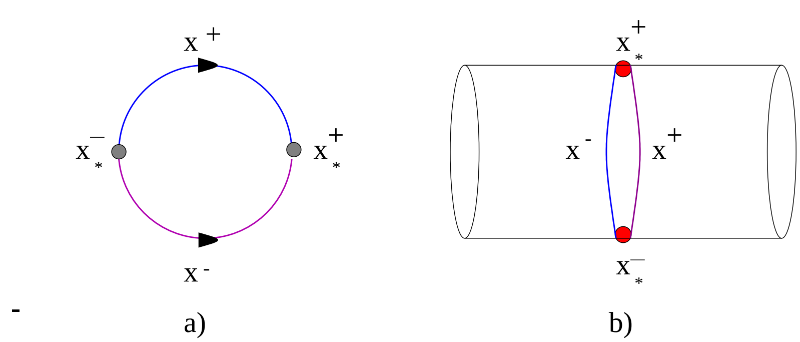

Figure 5: The two solitons of the $\mathbb{P}^1$ model.

such that their trajectories in the x-plane lie on two half-circles as shown in Fig. 5(a).

Since x is a $\mathbb{C}^\times$ coordinate we can consider the x-plane as a cylinder. Soliton trajectories on the cylinder are shown in Fig. 5(b). This description is useful in determining the intersection numbers of middle dimensional cycles. As described in the previous section the number of solitons between two critical points is given by the intersection number of middle dimensional cycles starting from the critical points. In our case there are two such cycles which are the preimages of two semi-infinite lines in the W-plane starting at the critical values as shown in Fig. 6(a). The preimage of these cycles on the cylinder is shown in Fig. 6(b). The cycles in the x-space intersect only if the lines in the W-plane

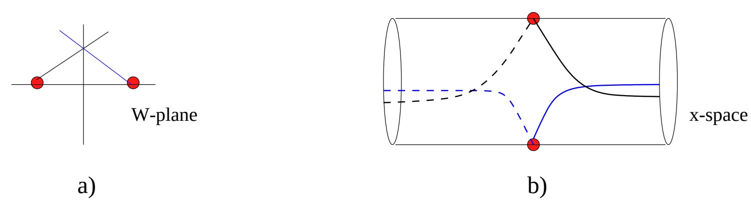

Figure 6: Intersecting lines in the W-plane and the corresponding intersecting cycles in the x-space.

intersect each other and the intersection number in this case is two.

We now turn to the study of solitons of the $\mathbb{P}^{N-1}$ sigma model. The LG theory mirror to the non-linear sigma model with $\mathbb{P}^{N-1}$ target space has superpotential [2]

$$W(X) = \sum_{k=1}^{N-1} X_k + \frac{\lambda}{X_1 \cdots X_{N-1}} . \tag{2.19}$$

This superpotential has N critical points given by

$$X_i^{(a)} = \mathrm{e}^{\frac{2\pi i a}{N}} \quad i = 1, \cdots, N-1 \; ; \quad a = 0, \cdots, N-1 \, , \tag{2.20}$$

with the critical values

$$w_a \equiv W(\vec{X}^{(a)}) = N \mathrm{e}^{\frac{2\pi i a}{N}} . \tag{2.21}$$

Here unlike the previous case of $\mathbb{P}^1$, to be able to solve for the preimage of a straight line, we will make an assumption about the soliton solution (For the case of $\mathbb{P}^2$ and its blowups we will also find another way to count the soliton numbers as will be discussed in section 8). Even though we will not justify this ansatz, the results we find are consistent with what is known based on tt^* equations. We assume that the soliton trajectory is determined by a function $f(t)$ such that

$$X_1 = X_2 = \cdots = X_k = f(t)^{N-k} \, , \; X_{k+1} = X_{k+2} = \cdots = X_N = f(t)^{-k} \, . \tag{2.22}$$

This parameterization of the solution satisfies the constraint $\prod_{i=1}^{N} X_i = 1$ by construction. With this ansatz the straight line equation in the W-plane becomes (for λ=1)

$$P(f) := kf^{N-k} + (N-k)f^{-k} = N(1 - t + te^{\frac{2\pi ik}{N}})\,, \tag{2.23}$$

where the right hand side is the straight line $w(t)$ starting from $w(0) = N$ and ending on $w(1) = N\mathrm{e}^{\frac{2\pi ik}{N}}$. Here we have chosen the parameter t running in the range $[0,1]$ that is linear in the W-plane. We are interested in the solutions which start at $t = 0$ from $X_i^{(0)}$ and end at $t = 1$ on $X_i^{(k)}$. This implies that $f(0)^{N-k} = f(0)^{-k} = 1$ and $f(1)^{N-k} = f(1)^{-k} = e^{\frac{2\pi ik}{N}}$. Thus the number of solitons which satisfy eq. (2.22) is given by the number of solutions to eq. (2.23) such that $f(0) = 1$ and $f(1) = e^{-\frac{2\pi i}{N}}$. We will show that there is only a single solution which satisfies these conditions.

Since $P'(1) = 0$ and $P''(1) \neq 0$, where prime denotes a differentiation with respect to f, only two trajectories start from $f = 1$. Thus it follows that the number of solutions is less than or equal to two. From eq. (2.23) it is clear that f can be real only at $t = 0$. Thus a trajectory cannot cross the real axis for $t > 0$. For t very close to zero one of the trajectories move into the upper half plane. Since the trajectory in the upper half plane cannot cross the real axis it cannot end on $e^{-\frac{2\pi ik}{N}}$ [1]. Thus there can be at most one solution.

To show that there actually exists a solution we will construct a solution whose image in the W-plane is homotopic to the straight line $w(t)$. Consider the function $f_*(t) = e^{-\frac{2\pi i}{N}t}$ where $t \in [0,1]$. Since

$$|P(f_*(t))| = |ke^{-2\pi it} + (N-k)| \leq |ke^{-2\pi it}| + (N-k) = N\,, \tag{2.24}$$

the image of $f_*(t)$ in the W-plane always lies inside the circle of radius N and only intersects the circle for $t = 0$ and $t = 1$ at $w = w_0$ and $w = w_k$ respectively. Thus the image is homotopic to the straight line $w(t)$ and therefore there exists a solution $f_0(t)$ homotopic to $f_*(t)$ with the required properties.

Since permuting the N coordinates among themselves does not change the superpotential, it follows that we can choose any k coordinates to be equal to f^{N-k} and the remaining $(N-k)$ coordinates equal to f^{-k}. Thus we see that there are $\binom{N}{k}$ solitons between the critical points $X_i^{(0)}$ and $X_i^{(k)}$ consistent with the ansatz of eq. (2.22). In fact this is the same number anticipated by the study of tt* equations [3]. Note that if the $\mathbb{P}^{N-1}$ has a round metric having $SU(N)$ symmetry, then the solitons should form

[1] Very close to $f = 1$ the two solutions are given by $f_\pm = 1 \pm \sqrt{\frac{2t(e^{\frac{2\pi ik}{N}}-1)}{k(N-k)}}$

representations of this group. In fact the permutations of X_i can be viewed as the Weyl group of the $SU(N)$, as is clear from the derivation of the mirror in this case [2]. It thus follows, given how the permutation acts on the solutions we have found, that in this case the solitons connecting vacua k units apart correspond to k fold anti-symmetric tensor product of the fundamental representation of $SU(N)$, a result which was derived from the large N analysis of this theory [17].

3 D-Branes in $\mathcal{N}=2$ Supersymmetric Field Theories

In this section, we study the $N=(2,2)$ supersymmetric field theory formulated on a 1+1 dimensional worldsheet with boundaries. We mainly consider supersymmetric sigma models and Landau-Ginzburg models. We find boundary conditions that preserve half of the (worldsheet) supersymmetry. (See [18, 19] for earlier works.) We define and compute the supersymmetric index of a theory on an interval. We also analyze the $\mathcal{N}=2$ boundary entropy defined as the pairing of the boundary states and the supersymmetric ground states.

3.1 The Supersymmetric Boundary Conditions

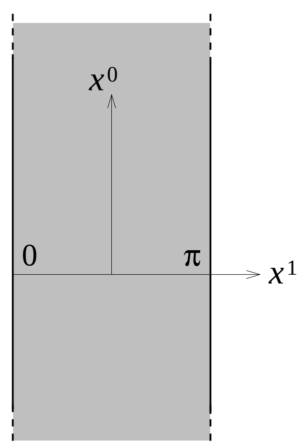

Figure 7: The strip $\mathbf{R}\times I$

Let us consider a supersymmetric sigma model on a Kähler manifold X of dimension n with a superpotential W. We denote the Kähler metric with respect to local complex coordinates z^i as $g_{i\bar{j}}$. We formulate the theory on the strip $\Sigma=\mathbf{R}\times I$ where I is an

interval $0 \leq x^1 \leq \pi$ and $\mathbf{R}$ is parametrized by the time coordinate x^0. Here, without any loss in generality we have fixed the size of the interval to a fixed length. Changing the length of the interval is equivalent to changing the parameters in the action according to the RG flow.

The action of the system is given by

$$
\begin{aligned}
S \;=\; \int_\Sigma \mathrm{d}^2x \Big\{ & -g_{i\bar{\jmath}}\partial^\mu\phi^i\partial_\mu\overline{\phi}^{\bar{\jmath}} + \frac{i}{2}g_{i\bar{\jmath}}\overline{\psi}_-^{\bar{\jmath}}(\overleftrightarrow{D}_0 + \overleftrightarrow{D}_1)\psi_-^i + \frac{i}{2}g_{i\bar{\jmath}}\overline{\psi}_+^{\bar{\jmath}}(\overleftrightarrow{D}_0 - \overleftrightarrow{D}_1)\psi_+^i \\
& - \frac{1}{4}g^{\bar{\jmath}i}\partial_{\bar{\jmath}}\overline{W}\partial_i W - \frac{1}{2}(D_i\partial_j W)\psi_+^i\psi_-^j - \frac{1}{2}(D_{\bar{\imath}}\partial_{\bar{\jmath}}\overline{W})\overline{\psi}_-^{\bar{\imath}}\overline{\psi}_+^{\bar{\jmath}} \\
& + R_{i\bar{k}j\bar{l}}\,\psi_+^i\psi_-^j\overline{\psi}_-^{\bar{k}}\overline{\psi}_+^{\bar{l}} \;\Big\},
\end{aligned}
\tag{3.1}
$$

where $\overline{\psi}^{\bar{\jmath}}\overleftrightarrow{D}_\mu\psi^i = \overline{\psi}^{\bar{\jmath}}(D_\mu\psi)^i - (D_\mu\overline{\psi})^{\bar{\jmath}}\psi^i$. See [2] for other notations. The above action is the same as the component expression of (2.1) up to a boundary term. We require the equations of motion for the fields $\phi^i, \psi_\pm^i$ to be local. This yields the following conditions on the boundary $\partial\Sigma$

$$
g_{IJ}\delta\phi^I\partial_1\phi^J = 0, \tag{3.2}
$$

$$
g_{IJ}(\psi_-^I\delta\psi_-^J - \psi_+^I\delta\psi_+^J) = 0, \tag{3.3}
$$

where $\phi^I, \psi_\pm^I$ and g_{IJ} are the components of the fields and the metric with respect to the real coordinates of the target space.

Under the supersymmetry transformation

$$
\delta\phi^i = \epsilon_+\psi_-^i - \epsilon_-\psi_+^i, \tag{3.4}
$$

$$
\delta\psi_+^i = i\overline{\epsilon}_-(\partial_0 + \partial_1)\phi^i + \epsilon_+F^i, \tag{3.5}
$$

$$
\delta\psi_-^i = -i\overline{\epsilon}_+(\partial_0 - \partial_1)\phi^i + \epsilon_-F^i, \tag{3.6}
$$

where

$$
F^i = -\frac{1}{2}g^{i\bar{\jmath}}\partial_{\bar{\jmath}}\overline{W} + \Gamma^i_{jk}\psi_+^j\psi_-^k, \tag{3.7}
$$

the action varies as

$$
\begin{aligned}
\delta S = \frac{1}{2}\int_{\partial\Sigma}\mathrm{d}x^0\Big\{ & \epsilon_+\Big(-g_{i\bar{\jmath}}(\partial_0+\partial_1)\overline{\phi}^{\bar{\jmath}}\psi_-^i + \frac{i}{2}\overline{\psi}_+^{\bar{\imath}}\partial_{\bar{\imath}}\overline{W}\Big) + \epsilon_-\Big(-g_{i\bar{\jmath}}(\partial_0-\partial_1)\overline{\phi}^{\bar{\jmath}}\psi_+^i - \frac{i}{2}\overline{\psi}_-^{\bar{\imath}}\partial_{\bar{\imath}}\overline{W}\Big) \\
& + \overline{\epsilon}_+\Big(g_{i\bar{\jmath}}\overline{\psi}_-^{\bar{\jmath}}(\partial_0+\partial_1)\phi^i + \frac{i}{2}\psi_+^i\partial_i W\Big) + \overline{\epsilon}_-\Big(g_{i\bar{\jmath}}\overline{\psi}_+^{\bar{\jmath}}(\partial_0-\partial_1)\phi^i - \frac{i}{2}\psi_-^i\partial_i W\Big)\Big\}.
\end{aligned}
\tag{3.8}
$$

If the boundary were absent, the action would be invariant under the full $(2,2)$ supersymmetry and the following four supercurrents would be conserved.

$$\begin{aligned}
G^0_\pm &= g_{i\bar{j}}(\partial_0 \pm \partial_1)\overline{\phi}^{\bar{j}}\psi^i_\pm \mp \frac{i}{2}\overline{\psi}^{\bar{\imath}}_\mp \partial_{\bar{\imath}}\overline{W}, \quad G^1_\pm = \mp g_{i\bar{j}}(\partial_0 \pm \partial_1)\overline{\phi}^{\bar{j}}\psi^i_\pm - \frac{i}{2}\overline{\psi}^{\bar{\imath}}_\mp \partial_{\bar{\imath}}\overline{W},\\
\overline{G}^0_\pm &= g_{i\bar{j}}\overline{\psi}^{\bar{j}}_\pm(\partial_0 \pm \partial_1)\phi^i \pm \frac{i}{2}\psi^i_\mp \partial_i W, \quad \overline{G}^1_\pm = \mp g_{i\bar{j}}\overline{\psi}^{\bar{j}}_\pm(\partial_0 \pm \partial_1)\phi^i + \frac{i}{2}\psi^i_\mp \partial_i W.
\end{aligned}$$

In what follows, we determine the boundary conditions on the fields $\phi^i, \psi^i_\pm$ that preserve half of the supersymmetry. We also wish to maintain the translation symmetry that maps the worldsheet boundary to itself, which is the time translation in the present set-up. There are essentially two possibilities for the unbroken supercharges [20];

(A) $Q = \overline{Q}_+ + \mathrm{e}^{i\alpha}Q_-$ and $Q^\dagger = Q_+ + \mathrm{e}^{-i\alpha}\overline{Q}_-$,

(B) $Q = \overline{Q}_+ + \mathrm{e}^{i\beta}\overline{Q}_-$ and $Q^\dagger = Q_+ + \mathrm{e}^{-i\beta}Q_-$.

Here $\mathrm{e}^{i\alpha}$ and $\mathrm{e}^{i\beta}$ are arbitrary phases. In both cases, the supercharges satisfy $\{Q, Q^\dagger\} = 2H$, up to a possible central term. The variation parameters for these supersymmetries are $\epsilon_- = \mathrm{e}^{i\alpha}\overline{\epsilon}_+$ for (A) while $\epsilon_- = -\mathrm{e}^{i\beta}\epsilon_+$ for (B). Conservation of the charges Q and $Q^\dagger$ requires that the spatial component of the corresponding currents vanish at the boundary $\partial\Sigma$: $\overline{G}^1_+ + \mathrm{e}^{i\alpha}G^1_- = G^1_+ + \mathrm{e}^{-i\alpha}\overline{G}^1_- = 0$ for (A), and $\overline{G}^1_+ + \mathrm{e}^{i\beta}\overline{G}^1_- = G^1_+ + \mathrm{e}^{-i\beta}G^1_- = 0$ for (B).

The conditions we are interested in are the ones associated with *D-branes* wrapped on a submanifold γ of X. Namely, we require the worldsheet boundary to be mapped to γ. In such a case, the derivative along the boundary $\partial_0\phi^I$ as well as an allowed variation $\delta\phi^I$ at the boundary must be tangent to γ. The locality condition (3.2) then tells that $\partial_1\phi^I$ must be normal to γ. The other condition (3.3) is satisfied if ψ^I_- and ψ^I_+ are related by an orthonormal transformation $\psi^I_- = M^I{}_J\psi^J_+$, $g_{IJ}M^I{}_K M^J{}_L = g_{KL}$. In fact, supersymmetry requires this and determines the matrix $M^I{}_J$, as we now show. For simplicity, we set the phases $\mathrm{e}^{i\alpha}$ and $\mathrm{e}^{i\beta}$ to be equal to 1; the general case can be easily recovered by $U(1)_V$ and $U(1)_A$ rotations. Then, both (A) and (B) contains an $N = 1$ subalgebra generated by the variations with parameter $\epsilon_+ = i\epsilon$ and $\epsilon_- = -i\epsilon$ where ϵ is real. Expressed in the real coordinates, the action varies as

$$\delta S = \frac{i\epsilon}{2}\int_{\partial\Sigma} \mathrm{d}x^0 \Big\{ -g_{IJ}\partial_0\phi^I(\psi^J_- - \psi^J_+) - g_{IJ}\partial_1\phi^I(\psi^J_- + \psi^J_+) - \frac{i}{2}(\psi^I_- + \psi^I_+)\partial_I(W - \overline{W})\Big\}. \quad (3.9)$$

Since $\partial_0\phi^I$ and $\partial_1\phi^i$ are tangent and normal to γ, the invariance of the action requires $i\epsilon(\psi^I_- - \psi^I_+)$ and $i\epsilon(\psi^I_- + \psi^I_+)$ to be normal and tangent to γ respectively. This means

that

$$\psi_-^I = \begin{cases} \psi_+^I & I: \text{ tangent} \\ -\psi_+^I & I: \text{ normal} \end{cases} \tag{3.10}$$

for a choice of coordinates that separates the tangent and normal directions. Furthermore, invariance of S requires $W - \overline{W}$ to be a constant along γ.

As we will see, A-type supersymmetry requires γ to be a middle dimensional Lagrangian submanifold whose image in the W-plane is a straight line, while B-type supersymmetry requires γ to be a holomorphic submanifold on which W is a constant.

A-Type Supersymmetry

We first consider A-type supersymmetry with the trivial phase $\mathrm{e}^{i\alpha} = 1$, which is generated by the variations with parameters $\epsilon_- = \overline{\epsilon}_+$ and $\overline{\epsilon}_- = \epsilon_+$. The bosonic fields ϕ^i transform as

$$\begin{aligned} \delta\phi^i &= \epsilon_+\psi_-^i - \overline{\epsilon}_+\psi_+^i \\ &= \epsilon_1(\psi_-^i - \psi_+^i) + i\epsilon_2(\psi_-^i + \psi_+^i), \end{aligned} \tag{3.11}$$

where ϵ_1 and ϵ_2 are the real and the imaginary parts of ϵ_+; $\epsilon_+ = \epsilon_1 + i\epsilon_2$. This shows that, for a real parameter ϵ, $\epsilon(\psi_-^i - \psi_+^i)$ and $i\epsilon(\psi_-^i + \psi_+^i)$ are the holomorphic components of tangent vectors of γ. On the other hand, $N = 1$ supersymmetry requires $i\epsilon(\psi_-^i - \psi_+^i)$ and $i\epsilon(\psi_-^i + \psi_+^i)$ are the holomorphic components of normal and tangent vectors of γ respectively. Thus, multiplication by $i = \sqrt{-1}$ on the holomorphic components sends tangent vectors to normal vectors and vice versa. Namely, the cycle γ must be a middle dimensional Lagrangian submanifold of X (where X is considered as a symplectic manifold defined by the Kähler form). The supersymmetry transformation of the tangent vector $\epsilon(\psi_-^i - \psi_+^i)$ is

$$\delta[\epsilon(\psi_-^i - \psi_+^i)] = 2i\epsilon\epsilon_1(\partial_0\phi^i) + 2i\epsilon\epsilon_2(i\partial_1\phi^i + F^i). \tag{3.12}$$

This must again be tangent to γ when $\delta\phi^i = 0$ (i.e. $\psi_\pm^i = 0$ for which $F^i = -\frac{1}{2}g^{i\bar{\jmath}}\partial_{\bar{\jmath}}\overline{W}$). We note that $i\epsilon\epsilon_1$ and $i\epsilon\epsilon_2$ are real parameters, $(i\epsilon\epsilon_i)^\dagger = -i\epsilon_i\epsilon = i\epsilon\epsilon_i$. Since $\partial_0\phi^i$, $i\partial_1\phi^i$ are both tangent to γ, the vector $v^i = g^{i\bar{\jmath}}\partial_{\bar{\jmath}}\overline{W}$ must be tangent to γ. This is consistent with the requirement of $N = 1$ supersymmetry that $W - \overline{W}$ must be a constant along γ, since the vector v^I annihilates $W - \overline{W}$

$$v^I\partial_I(W - \overline{W}) = |\partial W|^2 - |\partial W|^2 = 0. \tag{3.13}$$

It is easy to see that under these conditions the action S is invariant under the full A-type supersymmetry with $\mathrm{e}^{i\alpha} = 1$ and also that no other condition is required.

It is easy to recover the phase $\mathrm{e}^{i\alpha}$ using the $U(1)_V$ symmetry which rotates the fermions as $\psi^i_\pm \to \mathrm{e}^{i\alpha/2}\psi^i_\pm$ and $\overline{\psi}^{\bar{\jmath}}_\pm \to \mathrm{e}^{-i\alpha/2}\overline{\psi}^{\bar{\jmath}}_\pm$ and the superpotential as $W \to \mathrm{e}^{-i\alpha}W$. The boundary conditions on the fermions (3.10) are rotated accordingly. The condition on the cycle is also rotated: *the cycle γ is a middle dimensional Lagrangian submanifold of X with respect to the Kähler form, whose image in the W-plane is a straight line in the $\mathrm{e}^{i\alpha}$-direction.*

The axial $U(1)$ R-symmetry is not broken by the boundary condition. Indeed, $\epsilon(\psi^i_- - \psi^i_+)$ and $i\epsilon(\psi^i_- + \psi^i_+)$, which are holomorphic component of tangent vectors to γ (for $\mathrm{e}^{i\alpha} = 1$), are rotated within themselves by the axial rotation $\psi^i_\mp \to \mathrm{e}^{\pm i\theta_A}\psi^i_\mp$. If not anomalous (i.e. if $c_1(X) = 0$), the axial R-charge is conserved and the spatial component of the current must vanish at the boundary, $J^1_A = 0$ at $\partial\Sigma$. We note that the conserved supercharges Q and $Q^\dagger$ have axial charge 1 and -1 respectively.

The basic example of a cycle satisfying this condition is the wave-front trajectory emanating from a critical point of the superpotential. We consider here the case $\mathrm{e}^{i\alpha} = 1$ for simplicity. Let $p_* \in X$ be a non-degenerate critical point of W and let us consider a wave-front trajectory γ_{p_*} emanating from p_* in the positive real direction. As discussed in section 2 this corresponds to the totality of all potential soliton solutions starting at p_* whose image in the W-plane is stretched along the positive real axis. We recall that the one parameter family of maps f_t generated by the vector field $v^i = g^{i\bar{\jmath}}\partial_{\bar{\jmath}}\overline{W}$ acts on γ_{p_*} and any point on it is mapped by f_t to p_* in the limit $t \to -\infty$. By definition, γ_{p_*} is of middle dimension and the image in the W-plane is a straight-line in the real direction. To see that γ_{p_*} is a Lagrangian submanifold of X with respect to the Kähler form $\omega = ig_{i\bar{\jmath}}\mathrm{d}z^i \wedge \mathrm{d}\overline{z}^{\bar{\jmath}}$, it is crucial to note that

$$i_v\omega = i(g_{i\bar{\jmath}}v^i\mathrm{d}\overline{z}^{\bar{\jmath}} - g_{i\bar{\jmath}}\mathrm{d}z^i\overline{v}^{\bar{\jmath}}) = i\mathrm{d}(\overline{W} - W), \tag{3.14}$$

and hence

$$\mathcal{L}_v\omega = \mathrm{d}i_v\omega + i_v\mathrm{d}\omega = 0. \tag{3.15}$$

Thus, ω is invariant under the diffeomorphisms f_t. Let V_1 and V_2 be tangent vectors of γ_{p_*} at any point. Since the Kähler form is f_t-invariant, $\omega(f_tV_1, f_tV_2) = (f_t^*\omega)(V_1, V_2)$ is independent of t. However, in the limit $t \to -\infty$, the vectors f_tV_i become the zero vector at p_*. Thus, we have shown $\omega(V_1, V_2) = 0$. Namely, γ_{p_*} is Lagrangian.

B-Type Supersymmetry

We next consider B-type supersymmetry with the phase $e^{i\beta} = 1$ which is generated by $\epsilon_- = -\epsilon_+$ and $\overline{\epsilon}_- = -\overline{\epsilon}_+$. The bosonic fields ϕ^i transform as

$$\delta\phi^i = \epsilon_+(\psi_-^i + \psi_+^i). \tag{3.16}$$

Since ϵ_+ is a complex parameter, this shows that the tangent space to γ is invariant under the multiplication by $i = \sqrt{-1}$ on the holomorphic components. Namely, the cycle γ must be a complex submanifold of X. The supersymmetry transformation of the tangent vector $\psi_-^i + \psi_+^i$ is $\delta(\psi_-^i + \psi_+^i) = -2i\overline{\epsilon}_+\partial_0\phi^i$ which is indeed tangent to γ. On the other hand, the normal vector $\psi_-^i - \psi_+^i$ transforms as

$$\delta(\psi_-^i - \psi_+^i) = 2i\overline{\epsilon}_+\partial_1\phi^i + \epsilon_+ g^{i\bar{\jmath}}\partial_{\bar{\jmath}}\overline{W}, \tag{3.17}$$

at $\psi_\pm^i = 0$ for which $\delta\phi^i = 0$. This must again be normal to γ. Since $\partial_1\phi^i$ is normal to γ, this requires that $n^i = g^{i\bar{\jmath}}\partial_{\bar{\jmath}}\overline{W}$ is also a normal vector to γ. Namely, for a tangent vector v^i we have

$$0 = g_{i\bar{\jmath}} v^i \overline{n}^{\bar{\jmath}} = v^i\partial_i W. \tag{3.18}$$

Thus, not only the imaginary part $W - \overline{W}$ but W itself must be a constant on γ. It is easy to see that under these conditions the action is invariant under the full B-type supersymmetry with $e^{i\beta} = 1$ and also that no other condition is required.

It is again easy to recover the phase $e^{i\beta}$ using the $U(1)_A$ symmetry which rotates the fermions as $\psi_\pm^i \to e^{\pm i\beta/2}\psi_\pm^i$ and $\overline{\psi}_\pm^{\bar{\jmath}} \to e^{\mp i\beta/2}\overline{\psi}_\pm^{\bar{\jmath}}$. The boundary conditions on the fermions (3.10) are rotated accordingly, but the condition on the cycle remains the same: *the cycle γ is a complex submanifold of X on which W is a constant.*

The vector $U(1)$ R-symmetry is not broken by the boundary condition. Indeed, the tangent vector $\epsilon(\psi_-^i + \psi_+^i)$ to γ (for $e^{i\beta} = 1$) is rotated by phase under the vector rotation $\psi_\mp^i \to e^{i\theta_V}\psi_\mp^i$ and hence remains tangent. If not broken by the superpotential, the vector R-charge is conserved and the spatial component of the current must vanish at the boundary, $J_V^1 = 0$ at $\partial\Sigma$. We note that the conserved supercharges Q and $Q^\dagger$ has the vector charge 1 and -1 respectively.

3.1.1 Inclusion of the B-field

We can deform the theory by adding the following term to the action (3.1)

$$\frac{1}{2}\int_\Sigma B_{IJ}\mathrm{d}\phi^I \wedge \mathrm{d}\phi^J, \tag{3.19}$$

where $B = \frac{1}{2}B_{IJ}\mathrm{d}x^I \wedge \mathrm{d}x^J$ is a closed two-form on the manifold X. This term alters the condition (3.2) of locality for the bosonic equations of motion as

$$\delta\phi^I(g_{IJ}\partial_1\phi^J + B_{IJ}\partial_0\phi^J) = 0, \tag{3.20}$$

but the condition (3.3) for the fermionic equations of motion remains the same.

We look for the boundary conditions associated with the D-branes wrapped on a cycle γ in X which preserves A-type or B-type supersymmetry. By definition and by the requirement (3.20), the bosonic fields must obey the boundary conditions

$$\begin{aligned} &g_{IJ}\partial_1\phi^J + B_{IJ}\partial_0\phi^J = 0, && I: \text{ tangent,} \\ &\partial_0\phi^I = 0, && I: \text{ normal,} \end{aligned} \tag{3.21}$$

where we have chosen the coordinates that separate the tangent and the normal directions. For invariance under the $N = 1$ supersymmetry generated by $\epsilon_+ = i\epsilon$ and $\epsilon_- = -i\epsilon$ with ϵ being real (which is contained in both (A) and (B) supersymmetries with the trivial phases), the following boundary conditions on the fermions are required:

$$\begin{aligned} &g_{IJ}(\psi_-^J - \psi_+^J) - B_{IJ}(\psi_-^J + \psi_+^J) = 0, && I: \text{ tangent,} \\ &\psi_-^I + \psi_+^I = 0, && I: \text{ normal.} \end{aligned} \tag{3.22}$$

This also guarantees the condition (3.3). We also obtain the condition that the imaginary part of W is a constant along γ.

Proceeding as in the case without B-field, we obtain the following conditions on the cycle γ for the A- and B-type supersymmetry to be preserved. We only state the conditions for the cases with the trivial phase $\mathrm{e}^{i\alpha} = \mathrm{e}^{i\beta} = 1$ since the generalization is clear.

A-type Supersymmetry

γ is a middle dimensional Lagrangian submanifold of X on which (not only the Kähler form but also) the B-field is annihilated, $B|\gamma = 0$. The image in the W-plane must be a straight line in the real direction.

B-type Supersymmetry

γ is a complex submanifold of X. B-field evaluated on the holomorphic tangent vectors to γ is zero, $(B|\gamma)^{(2,0)} = 0$. Also, W must be a constant on γ.

3.1.2 Coupling to the Gauge Fields on the Branes

We can couple the worldsheet boundaries to the gauge fields on the branes. In the case of the strip $\Sigma = \mathbf{R} \times I$, this corresponds to adding to the action (3.1) the terms

$$\int_{\partial\Sigma} A_M \mathrm{d}\phi^M = \int_{x^1=\pi} \mathrm{d}x^0 \partial_0 \phi^{M_b} A^{(b)}_{M_b} - \int_{x^1=0} \mathrm{d}x^0 \partial_0 \phi^{M_a} A^{(a)}_{M_a}, \tag{3.23}$$

where $A^{(a)}$ and $A^{(b)}$ are the $U(1)$ gauge fields on the branes γ_a and γ_b on which the left and the right boundaries end. (We use $M, N, \ldots$ for coordinate indices on the branes.) If the left and the right boundaries are coupled to the same gauge field A that extends to the whole target space X, the boundary terms (3.23) can be written as

$$\frac{1}{2} \int_{\Sigma} F_{IJ} \mathrm{d}\phi^I \wedge \mathrm{d}\phi^J \tag{3.24}$$

where $F = \frac{1}{2} F_{IJ} \mathrm{d}x^I \wedge \mathrm{d}x^J$ is the curvature of the gauge field, $F = \mathrm{d}A$. Thus, in this case, we can treat the gauge field coupling in the same way as the coupling to the B-field. In particular, we have the local equations of motion and $N = 1$ supersymmetry by imposing the boundary conditions (3.21) and (3.22) with $B \to F$. When the cycle γ is a middle dimensional Lagrangian submanifold of X whose image in the W-plane is a straight line, the theory is invariant under A-type supersymmetry if the gauge field is flat on the cycle, $F|\gamma = 0$. When the cycle γ is a complex submanifold of X on which W is a constant, the theory is invariant under B-type supersymmetry if the gauge field has a $(1, 1)$ curvature on γ, $(F|\gamma)^{(2,0)} = 0$, namely, if $A|\gamma$ determines a holomorphic line bundle on γ.

The conclusion obtained above remains valid even if the left and the right boundary components are coupled to different gauge fields that are defined only on the branes. Thus, *one can deform the A-type supersymmetric theory by flat gauge fields on γ while B-type supersymmetric theory can be deformed by holomorphic line bundles on γ.*

An Alternative Formulation for B-type D-branes

For A-type D-branes with a flat gauge field, the boundary condition given by (3.21) and (3.22) (with $B_{IJ} \to F_{IJ}$) is the same as the standard one

$$\begin{aligned} &\partial_1 \phi^I = 0, \quad \psi^I_- - \psi^I_+ = 0, \quad I: \text{ tangent}, \\ &\partial_0 \phi^I = 0, \quad \psi^I_- + \psi^I_+ = 0 \quad\; I: \text{ normal}. \end{aligned} \tag{3.25}$$

However, it is in general different from (3.25) for B-type D-branes where the gauge field is not necessarily flat. There is actually an alternative formulation for B-type D-branes

where we still impose the standard boundary condition (3.25). It is easy to see that (with $\psi^I := (\psi_+^I + \psi_-^I)/2$)

$$\int\limits_{\partial\Sigma} \mathrm{d}x^0 \left\{ \partial_0\phi^M A_M + iF_{MN}\psi^M\psi^N \right\} \tag{3.26}$$

is invariant by itself under the B-type supersymmetry if the gauge field is holomorphic, $F_{mn} = F_{\bar{m}\bar{n}} = 0$. Thus, instead of (3.23), one can add the boundary term (3.26) without breaking the B-type supersymmetry of the bulk action (3.1) which holds under (3.25). We note however that the equations of motion for the fields ϕ^I, $\psi_\pm^I$ are modified by boundary terms. This formulation was used in [21–23] to study the fluctuation of the target space gauge fields in string theory.

Non-Abelian Gauge Fields

One can generalize the above analysis to non-abelian $U(k)$ gauge group [22, 24]. In this case, the path-integral weight $\exp(iS)$ is accompanied by the matrix factors

$$P_{\partial\Sigma} \exp\left(i \int_{\partial\Sigma} \mathrm{d}x^0 \left\{ \partial_0\phi^M A_M + iF_{MN}\psi^M\psi^N \right\} \right) \tag{3.27}$$

where $P_{\partial\Sigma}$ is (the product of) the path-ordering along the boundary $\partial\Sigma$. Under the standard boundary condition (3.25), the weight (3.27) is invariant under A-type supersymmetry if A is flat, $F_{MN} = 0$, while it is invariant under B-type supersymmetry if A is holomorphic, $F_{mn} = F_{\bar{m}\bar{n}} = 0$.

3.2 Supersymmetric Ground States

As in any supersymmetric field theory, in the theory on the segment $I = [0, \pi]$ with the boundary condition that preserves A or B-type supersymmetry, one can define the supersymmetric index $\mathrm{Tr}(-1)^F$ which is invariant under deformations of the theory. We denote this index as

$$I(a, b) = \mathrm{Tr}\,(-1)^F, \tag{3.28}$$

where a and b are the boundary conditions at the left and the right boundaries.[1] We shall compute this index in the two basic examples; A-type D-branes in Landau-Ginzburg models and B-type D-branes in non-linear sigma models. Actually, in the LG models (not only the index but also) the complete spectrum of supersymmetric ground states can be determined. This can also be done for non-linear sigma models under a certain condition on the cohomology of the gauge bundles. For simplicity, we set the phases $\mathrm{e}^{i\alpha} = 1$ and $\mathrm{e}^{i\beta} = 1$.

[1] The index (3.28) for supersymmetric D-branes in Calabi-Yau manifolds was studied in [30, 32].

3.2.1 Landau-Ginzburg Models

Let us consider a LG model with superpotential W. We assume that the bosonic potential $U = |\partial W|^2$ diverges at infinity in the configuration space X. We also assume that there is no non-trivial B field and we will not consider coupling to gauge field on the branes. Let a and b be two non-degenerate critical points of W. We consider the wave-front trajectories γ_a and γ_b emanating from a and b in the positive real direction in the W-plane. We assume for now that the half-lines $W(\gamma_a)$ and $W(\gamma_b)$ are separated in the imaginary direction, and there is no other critical values of W between them. We consider the theory on $[0,\pi]$ where the left boundary $x^1 = 0$ is mapped to γ_a and the right boundary $x^1 = \pi$ is mapped to γ_b. For the boundary condition described earlier, the theory is invariant under A-type supersymmetry generated by the supercharges $Q = \overline{Q}_+ + Q_-$ and $Q^\dagger = Q_+ + \overline{Q}_-$, which are expressed as

$$\begin{aligned} Q &= \sqrt{2}\int_0^\pi \mathrm{d}x^1 \left\{ \overline{\psi}_+^{\bar{\jmath}}\Big(g_{i\bar{\jmath}}(\partial_0+\partial_1)\phi^i + \frac{i}{2}\partial_{\bar{\jmath}}\overline{W}\Big) + \psi_-^i\Big(g_{i\bar{\jmath}}(\partial_0-\partial_1)\overline{\phi}^{\bar{\jmath}} + \frac{i}{2}\partial_i W\Big)\right\}, \\ Q^\dagger &= \sqrt{2}\int_0^\pi \mathrm{d}x^1 \left\{ \overline{\psi}_-^{\bar{\jmath}}\Big(g_{i\bar{\jmath}}(\partial_0-\partial_1)\phi^i - \frac{i}{2}\partial_{\bar{\jmath}}\overline{W}\Big) + \psi_+^i\Big(g_{i\bar{\jmath}}(\partial_0+\partial_1)\overline{\phi}^{\bar{\jmath}} - \frac{i}{2}\partial_i W\Big)\right\}. \end{aligned} \tag{3.29}$$

The supercharges Q and $Q^\dagger$ are nilpotent and satisfy the anti-commutation relation

$$\{Q, Q^\dagger\} = 4(H + \varDelta\mathrm{Im}W),$$

where $\varDelta\mathrm{Im}W = \mathrm{Im}W(b) - \mathrm{Im}W(a)$ is the separation of the two half-lines in the imaginary direction. We shift the definition of the Hamiltonian as $\widetilde{H} = H + \varDelta\mathrm{Im}W$ so that the supersymmetry algebra takes the standard form $\{Q, Q^\dagger\} = 4\widetilde{H}$. Since $\varDelta\mathrm{Im}W$ is a constant, this is done simply by the shift of the action

$$\widetilde{S} = S - \int\limits_{x^1=\pi} \mathrm{d}x^0\,\mathrm{Im}W + \int\limits_{x^1=0} \mathrm{d}x^0\,\mathrm{Im}W. \tag{3.30}$$

The index can be defined by $I(a,b) = \mathrm{Tr}\,(-1)^F \mathrm{e}^{-\beta\widetilde{H}}$, and only the ground states with energy $\widetilde{H} = 0$ can contribute to this. One can see from the expressions (3.29) that $Q = Q^\dagger = 0$ for a static configuration such that

$$\partial_1\phi^i = -\frac{i}{2}g^{i\bar{\jmath}}\partial_{\bar{\jmath}}\overline{W}. \tag{3.31}$$

Namely, the supersymmetry is classically preserved for a static configuration that goes from γ_a to γ_b, straight down in the negative imaginary direction of the W-plane. Such a configuration would indeed have $H = -\varDelta\mathrm{Im}W$ or $\widetilde{H} = 0$ and satisfy the required

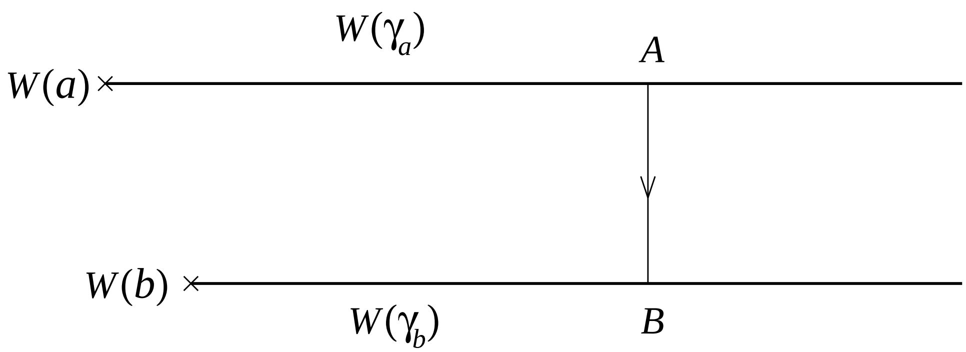

Figure 8: The image of D-branes in the W-plane and a path between them

boundary condition. We note that there is no such configuration if $\mathrm{Im}W(a) < \mathrm{Im}W(b)$. In such a case $I(a,b) = 0$.

Now let us compute the index. We are considering the situation as depicted in Figure 8 where the arrowed line from A to B is a straight segment in the negative imaginary direction of the W-plane. We consider the wave-front at the point B along the straight line from $W(b)$ and another wave-front at B along the broken segment starting from $W(a)$ and bending at the point A. From the general theory of singularities, the two wave-fronts have intersection number $\Delta_a \circ \Delta_b$, the same as the soliton number between a and b. This means that there are $\Delta_a \circ \Delta_b$ paths from γ_a to γ_b that maps to the straight segment from A to B in the W-plane. Since this holds for any starting point A, there are $\Delta_a \circ \Delta_b$ families of such paths parametrized by $w_1 := \mathrm{Re}A = \mathrm{Re}B$. It may appear that there are infinitely many solutions to (3.31) and therefore infinite degeneracy of supersymmetric ground states. However, we note that the length of x^1 that is required to go from γ_a to γ_b depends on each path and does not necessarily coincide with π. The required length of x^1 for each path P is given by

$$\Delta x^1 = \left| 2 \int_P \frac{\mathrm{dIm}W}{|\partial W|^2} \right| . \tag{3.32}$$

Only the path with $\Delta x^1 = \pi$ defines a classical supersymmetric ground state. If the starting point A or the end point B is the critical value $W(a)$ or $W(b)$, the required length is infinity $\Delta x^1 = +\infty$. In the massive theory where the bosonic potential $U = |\partial W|^2$ diverges at infinity, Δx^1 approaches zero when $w_1 = \mathrm{Re}A$ goes to infinity. Thus, for each of the $\Delta_a \circ \Delta_b$ families, Δx^1 is roughly a decreasing function as a function of w_1. If it is a monotonic function, the function (3.32) cut through $\Delta x^1 = \pi$ exactly once and hence the contribution to the index of that family is 1. However, one may encounter a family

where it cuts through $\varDelta x^1 = \pi$ more than once as depicted in Figure 9. In such a case,

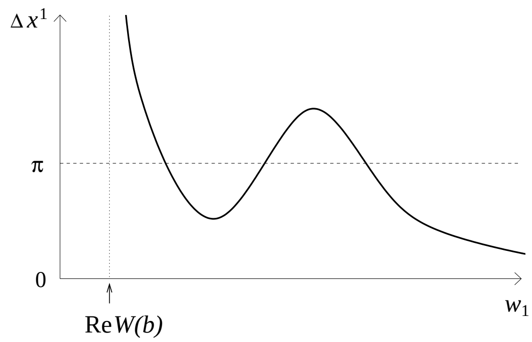

Figure 9: A Graph of $\varDelta x^1$ as a function of $w_1 = \mathrm{Re}A = \mathrm{Re}B$. Corresponding to the situation in Figure 8, the left end is set at $w_1 = \mathrm{Re}W(b)$.

we make use of the fact that the index is invariant under the deformation of the theory. In particular, we can rescale the superpotential as $W \to \mathrm{e}^t W$. This changes the function (3.32) as $\varDelta x^1 \to \mathrm{e}^{-t}\varDelta x^1$. For an appropriate choice of e^t one can make $\mathrm{e}^{-t}\varDelta x^1$ to cut through π exactly once. Thus, in any case, the contribution to the Witten index is 1 for each family. Thus, the total index is given by

$$I(a,b) = \begin{cases} \Delta_a \circ \Delta_b & \text{if } \ \mathrm{Im}W(a) > \mathrm{Im}W(b), \\ 0 & \text{if } \ \mathrm{Im}W(a) < \mathrm{Im}W(b). \end{cases} \tag{3.33}$$

In the case $a = b$ there is of course a unique classical supersymmetric configuration: $\phi^I(x^1) = a^I$ (constant along x^1). Thus, we have

$$I(a,a) = 1. \tag{3.34}$$

It is easy to generalize the above analysis to the case where there are critical values between $W(\gamma_a)$ and $W(\gamma_b)$. Let us consider the simplest case where there is one critical value $W(c)$. Then, the number of paths from γ_a to γ_b depends on whether the image in the W-plane is on the left or right of $W(c)$, i.e. whether $w_1 < \mathrm{Re}W(c)$ or $w_1 > \mathrm{Re}W(c)$; we denote these numbers as $(\Delta_a \circ \Delta_b)_<$ and $(\Delta_a \circ \Delta_b)_>$ respectively. Some paths on the left smoothly continue to the right of $W(c)$. However, some others hit the critical point c at $w_1 = \mathrm{Re}W(c)$ where $\varDelta x^1$ blows up to infinity. The length $\varDelta x^1$ is bounded from below by a positive value for the paths on the left of $W(c)$. Thus, by rescaling the superpotential if necessary, we can have a situation where the solution with $\varDelta x^1 = \pi$ exists only on the right of $W(c)$. Thus, the index is given by $I(a,b) = (\Delta_a \circ \Delta_b)_>$. It

is obvious how to generalize this argument to the case where there are more than one critical values in the region between $W(\gamma_a)$ and $W(\gamma_b)$. The index is still given by (3.33) where it is understood that $\Delta_a \circ \Delta_b$ stands for the intersection number of the wavefronts corresponding to the paths in the W-plane that meet with each other on the right of the right-most critical value. Note that this is the same as the "intersection number" of γ_a and γ_b defined in the previous section; namely, the number $\#(\gamma_a \cap \gamma_b')$ where γ_b' is obtained by tilting γ_b with an infinitesimal positive angle against the real axis in the W-plane.

The asymmetry in $I(a,b)$ under the interchange of a and b has an interesting interpretation, as we will discuss later in this paper. The mirror version of the same asymmetry is discussed in the next subsection, for holomorphic D-branes on sigma models.

One might be interested in exactly how many supersymmetric ground states are there. If the function (3.32) for a family of paths cut through $\mathit{\Delta} x^1 = \pi$ exactly once, there is of course one ground state from that family. However, one may find a family where the graph of $\mathit{\Delta} x^1$ looks like Figure 9. In such a case, extra pairs of states may potentially become supersymmetric ground states (though do not contribute to the index). To see whether this is possible or not, we show that, under a certain assumption, the system under consideration is nothing but the supersymmetric quantum mechanics considered in [25, 27] applied to the infinite-dimensional space of paths. In [25], the system with Hamiltonian $H = \frac{1}{2}p^2 + \frac{1}{2}(h'(x))^2 + \frac{1}{2}h''(x)(\psi\overline{\psi} - \overline{\psi}\psi)$ is considered where $h(x)$ is a real valued function. This system possesses supersymmetry generated by $Q = \sqrt{2}\overline{\psi}(p+ih'(x))$ and $Q^\dagger = \sqrt{2}\psi(p - ih'(x))$ which satisfy $\{Q, Q^\dagger\} = 4H$. It is shown that there is a single supersymmetric ground state as long as $h'(x)$ cuts through $h'(x) = 0$ odd times (no matter how many), but there is none if it cuts through $h'(x) = 0$ even times. The analysis is extended in [27] to the supersymmetric quantum mechanics on a Riemannian manifold deformed by a Morse function h. In particular, the supersymmetric ground states are realized as cohomology classes of a cochain complex constructed from the critical points of h with the grading determined by the Morse index. Let $\Omega_{ab}X$ be the space of paths $[0,\pi] \to X$ from γ_a to γ_b, with the boundary condition that the derivatives at $x^1 = 0$ and π are normal to γ_a and γ_b respectively. From the inspection of the supercharges (3.29), we see that the present system is nothing but the supersymmetric quantum mechanics on $\Omega_{ab}X$ deformed in the same way as [25, 27] if there is a function h on $\Omega_{ab}X$ such that $\delta h/\delta\phi^i = ig_{i\bar{\jmath}}\partial_1\overline{\phi}^{\bar{\jmath}} + \frac{1}{2}\partial_i W$, and $\delta h/\delta\overline{\phi}^{\bar{\jmath}} = -ig_{i\bar{\jmath}}\partial_1\phi^i + \frac{1}{2}\partial_{\bar{\jmath}}\overline{W}$; in other words

$$\delta h = \int_0^\pi \mathrm{d}x^1 \left\{ \omega_{IJ}\delta\phi^I\partial_1\phi^J + \delta\phi^I\partial_I \mathrm{Re}W \right\}, \tag{3.35}$$

where ω is the Kähler form of X, $\omega = ig_{i\bar{\jmath}}\mathrm{d}z^i \wedge \mathrm{d}\overline{z}^{\bar{\jmath}}$. If we choose a base point ϕ_0 of (each

connected component of) $\Omega_{ab}X$, one can "define" a function

$$h[\phi] = \frac{1}{2}\int\limits_{[0,1]\times[0,\pi]} \widehat{\phi}^*\omega \;+\; \int_0^\pi \mathrm{d}x^1 \mathrm{Re}W, \tag{3.36}$$

where $\widehat{\phi}(s, x^1)$ is a homotopy in $\Omega_{ab}X$ connecting ϕ and ϕ_0, namely a map $\widehat{\phi} : [0,1]\times[0,\pi] \to X$ such that $\widehat{\phi}(0,x^1) = \phi_0(x^1)$, $\widehat{\phi}(1,x^1) = \phi(x^1)$, and obeying the Dirichlet/Neumann boundary condition at $x^1 = 0$ and $x^1 = \pi$. Using the fact that γ_a and γ_b are Lagrangian and recalling the boundary condition that $\partial_1\phi^I|_{\partial\Sigma}$ is normal to the brane, it is easy to see that $h[\phi]$ is invariant under a small variation of the homotopy and that it satisfies (3.35) for the variation of ϕ. For a large change of homotopy, using the fact that the cycles γ_a and γ_b are simply connected, we can show that $h[\phi]$ changes by $\frac{1}{2}\int_C \omega$ where C is a two-cycle in X. We discard this subtlety by focusing our attention only to those cases where $\int_C \omega = 0$ for all (compact) two-cycle C. We also assume $c_1(X) = 0$. These assumptions hold in the class of models we consider later in this paper. Then, (3.36) is independent of the choice of homotopy and becomes a well-defined function on $\Omega_{ab}X$. Under the assumption $c_1(X) = 0$, $U(1)$ axial R-charge is conserved and the supercharge Q has charge 1. Thus, we can define the cochain complex as in [27] graded by the axial R-charge.[2] The coboundary operator is defined by counting the number of instantons connecting different critical points of $h[\phi]$. We note that an instanton connecting critical paths ϕ_1 and ϕ_2 is a configuration $\phi(\tau, x^1)$ such that $\phi(-\infty, x^1) = \phi_1(x^1)$, $\phi(+\infty, x^1) = \phi_2(x^1)$ and satisfying the following equation

$$\frac{\partial\phi^i}{\partial\tau} = -i\frac{\partial\phi^i}{\partial x^1} + \frac{1}{2}g^{i\bar{j}}\partial_{\bar{j}}\overline{W}. \tag{3.37}$$

We denote the corresponding cohomology groups as

$$\mathrm{HF}^p_W(\gamma_a, \gamma_b), \tag{3.38}$$

where the grading p is given by the axial R-charge of the ground states. This is a Landau-Ginzburg generalization of the Floer homology group [28]. Under the rescaling of the superpotential $W \to \mathrm{e}^t W$, the supercharge simply changes as $Q \to \mathrm{e}^{-\Delta h} Q \mathrm{e}^{\Delta h}$ where $\Delta h = (\mathrm{e}^t - 1)\int_0^\pi \mathrm{d}x^1 \mathrm{Re}W$. Since multiplication by $\mathrm{e}^{-\Delta h}$ is well-defined, the cohomology

[2] There is a subtlety for defining the grading (Morse index) from the fact that the Hessian of $h[\phi]$ has unbounded spectrum. However, one can regularize it, up to an additive constant, by the index of the corresponding Dirac-type operator which is well-defined under the assumption that $c_1(X) = 0$ (this is related to the conservation of the axial R-charge). We set the ground of the grading so that it is 0 for the critical path which is unique in the family (unlike Figure 9).

is invariant under this rescaling.[3] Then, we see that for $\mathrm{Im}W(a) > \mathrm{Im}W(b)$

$$\mathrm{HF}^p_W(\gamma_a,\gamma_b) = \begin{cases} \mathbf{R}^{|\Delta_a\circ\Delta_b|} & p=0, \\ 0 & p\neq 0. \end{cases} \tag{3.39}$$

In the situation as in Figure 9, we see that pairs of classical supersymmetric ground states are lifted by an instanton effect. Those states have very small (but positive) energies.

It would be interesting to generalize the above consideration to the case where there are two-cycles with $\int_C\omega\neq 0$ and also to the case where $c_1(X)\neq 0$. In some of the latter cases, we expect that the cohomology (3.38) is not graded by integers, but by some cyclic group.

3.2.2 Sigma Models

The other example we consider is the supersymmetric sigma model on X with trivial superpotential $W=0$ where the D-branes are wrapped totally on X. We couple the left and the right boundaries to $U(1)$ gauge fields $A^{(a)}$ and $A^{(b)}$ respectively that define holomorphic line bundles E_a and E_b on X. We use the formulation where the boundary term is given by (3.26) and the boundary condition is the standard one (3.25) (where there is no normal direction in the present case). The theory is invariant under B-type supersymmetry generated by $Q=\overline{Q}_+ + \overline{Q}_-$ and $Q^\dagger = Q_+ + Q_-$. Since the boundary term (3.26) includes the time derivatives of the fields, the Noether charges are modified. Thus, the supercharge Q is expressed as

$$\begin{aligned} Q \;=\; \sqrt{2}\bigg(&\int_0^\pi \mathrm{d}x^1 \left\{ g_{i\bar{j}}(\overline{\psi}^{\bar{j}}_+ + \overline{\psi}^{\bar{j}}_-)\partial_0\phi^i - g_{i\bar{j}}(\overline{\psi}^{\bar{j}}_+ - \overline{\psi}^{\bar{j}}_-)\partial_1\phi^i \right\} \\ &+ (\overline{\psi}^{\bar{j}}_+ + \overline{\psi}^{\bar{j}}_-)A^{(b)}_{\bar{j}}\Big|_{x^1=\pi} - (\overline{\psi}^{\bar{j}}_+ + \overline{\psi}^{\bar{j}}_-)A^{(a)}_{\bar{j}}\Big|_{x^1=0} \bigg). \end{aligned} \tag{3.40}$$

For the purpose of computing the index, we can focus on the zero modes (x^1-independent modes). Then from the boundary condition, the left and the right fermionic zero modes are related as $\psi^i_{-0} = \psi^i_{+0}$ and $\overline{\psi}^{\bar{\imath}}_{-0} = \overline{\psi}^{\bar{\imath}}_{+0}$. We can identify the quantum mechanical Hilbert space as the space of sections on the bundle

$$\Big(\bigwedge T^{*(0,1)}X\Big)\otimes E_a^*\otimes E_b, \tag{3.41}$$

[3] Alternatively, as in [28] one may construct a cochain homotopy equivalence of the reduced complexes introduced above.

on which the fermionic zero modes act as[4]

$$\frac{1}{\sqrt{2}}(\overline{\psi}^{\bar{\imath}}_{+0} + \overline{\psi}^{\bar{\imath}}_{-0}) \quad \longleftrightarrow \quad \mathrm{d}\overline{z}^{\bar{\imath}}\wedge, \tag{3.42}$$

$$\frac{1}{\sqrt{2}} g_{i\bar{\jmath}}(\psi^i_{+0} + \psi^i_{-0}) \quad \longleftrightarrow \quad i_{\partial/\partial\overline{z}^{\bar{\jmath}}}. \tag{3.43}$$

Then the supercharge Q corresponds to the Dolbeault operator on the bundle $E_a^* \otimes E_b$:

$$Q \;\leftrightarrow\; 2\overline{\partial}_A = 2\mathrm{d}\overline{z}^{\bar{\imath}}\left(\partial_{\bar{\imath}} + A^{(b)}_{\bar{\imath}} - A^{(a)}_{\bar{\imath}}\right). \tag{3.44}$$

Thus, the Witten index, which is defined as the index of Q operator, is equal to the index of this Dolbeault operator. By the standard index theorem, we obtain

$$I(a,b) = \chi(E_a, E_b) := \int_X \mathrm{ch}(E_a^* \otimes E_b)\mathrm{Td}(X), \tag{3.45}$$

where $\mathrm{Td}(X)$ is the total Todd class of the tangent bundle of X which are given by polynomials of the Chern classes (see e.g. [26]). It is easy to extend this analysis to the case where the bundles E_a and E_b have higher ranks. The conclusion remains the same as (3.45).

In general, the index (3.45) is not symmetric nor anti-symmetric under the exchange of a and b. This is related by mirror symmetry, as we will discuss later, with the fact noted earlier, that supersymmetric index $I(a,b)$ for Lagrangian D-branes in LG models is neither symmetric nor anti-symmetric. However, since odd Todd classes are divisible by the first Chern class of X [26], for a Calabi-Yau manifold $\mathrm{Td}(X)$ is a sum of $4k$-forms. Under the exchange $E_a^* \otimes E_b \to E_b^* \otimes E_a$ the Chern character changes by sign flip in the $(4k+2)$-form components. Thus, for a Calabi-Yau manifold of dimension n, the index $I(a,b)$ is symmetric for even n and anti-symmetric for odd n under the exchange of a and b.

One can actually obtain an upper bound on the number of supersymmetric ground states, using a technique of section 3 of [27]. The cohomology of operator Q is actually invariant under the rescaling $\partial_1\phi^i \to \mathrm{e}^t\partial_1\phi^i$, since it is done by conjugation by e^{tP} where P is an operator counting the number of fermions of combination $\overline{\psi}^{\bar{\jmath}}_+ - \overline{\psi}^{\bar{\jmath}}_-$. This means that the cohomology group is independent of the width of the strip in the x^1 direction,

[4]Unlike in the closed string case [29], we do not have the factor $\bigwedge TX^{(1,0)}$ in (3.41) nor

$$\frac{1}{\sqrt{2}} g_{i\bar{\jmath}}(\overline{\psi}^{\bar{\jmath}}_{+0} - \overline{\psi}^{\bar{\jmath}}_{-0}) \leftrightarrow (\partial/\partial z^i)\wedge, \quad \frac{1}{\sqrt{2}}(\psi^i_{+0} - \psi^i_{-0}) \leftrightarrow i_{\mathrm{d}z^i},$$

because $\overline{\psi}^{\bar{\jmath}}_{+0} - \overline{\psi}^{\bar{\jmath}}_{-0} = 0$ and $\psi^i_{+0} - \psi^i_{-0} = 0$ from the boundary condition.

as long as it is finite. A zero energy state should remain a zero energy state as we make the strip thinner and thinner. In particular, a ground state must correspond to a ground state of the quantum mechanics of the zero modes. In this way we obtain the upper bound on the number of ground states. The ground states of the quantum mechanics are the cohomology classes of the Dolbeault complex $\Omega^{0,p}(X, E_a^* \otimes E_b)$ with (3.44) as the coboundary operator. Thus, the quantum mechanical ground states are given by the Dolbeault cohomology

$$\mathrm{H}^{0,p}(X, E_a^* \otimes E_b). \tag{3.46}$$

We note that the vector R-symmetry, for which Q has charge 1, is not broken in the bulk nor by the boundary condition. Thus, the grading p of the cohomology group (3.46) is the same as the vector R-charge. The group (3.46) gives us an upper bound on the number of supersymmetric ground states, but we do not have a lower bound in general (the argument in section 3 of [27] does not apply here)[5]. However, if the cohomology is non-vanishing only for even p (or only for odd p), the cohomology group (3.46) is indeed the same as the space of supersymmetric ground states. Later in this paper, we will consider a certain set of bundles such that the cohomology (3.46) vanishes except $p = 0$ and hence it can be identified as the space of ground states.

3.3 The Boundary States

Let us consider a Euclidean quantum field theory formulated on a Riemann surface Σ with boundary circles. We choose an orientation of each component S^1 of the boundary and we call it an *incoming* (resp. *outgoing*) component if the 90° rotation of the positive tangent vector of S^1 (with respect to the orientation of Σ) is an inward (outward) normal vector at the boundary. We choose the metric on Σ such that it is a flat cylinder near each boundary component. Suppose Σ has a single outgoing boundary, $S^1 = \partial\Sigma$. The partition function on Σ depend on the boundary condition a on the fields at $\partial\Sigma$ and we denote it by $Z^a(\Sigma)$. On the other hand, the path-integral over the fields on Σ defines a state $|\Sigma\rangle$ that belongs to the quantum Hilbert space $\mathcal{H}_{S^1}$ at the boundary circle. We

[5]This is analogous to the situation in the sigma model on a worldsheet without boundary. B-type supercharge yields Dolbeault complex with the coefficient $\bigwedge TX^{(1,0)}$ in the zero mode approximation. This indeed has the correct Witten index

$$\sum_{p,q}(-1)^{p+q}\dim \mathrm{H}^{0,p}(X, \wedge^q TX^{(1,0)}) = \pm\chi(X).$$

However, the cohomology group $\oplus_{p,q}\mathrm{H}^{0,p}(X, \bigwedge^q TX^{(1,0)})$ itself is in general larger than the space of ground states which we know to be $\oplus_{p,q}\mathrm{H}^{q,p}(X)$, unless X is a Calabi-Yau manifold.

define the *boundary state* $\langle a|$ corresponding to the boundary condition a by the property

$$Z^a(\Sigma) = \langle a|\Sigma\rangle. \tag{3.47}$$

If Σ has a single incoming boundary $\partial\Sigma = S^1$, we have a state $\langle\Sigma|$ that belongs to the dual space $\mathcal{H}_{S^1}^\dagger$. For a boundary condition b at S^1, we define the boundary state $|b\rangle$ by

$$Z_b(\Sigma) = \langle\Sigma|b\rangle, \tag{3.48}$$

where $Z_b(\Sigma)$ stands for the partition function on Σ with the boundary condition b. In general, the boundary state $\langle a|$ (resp. $|b\rangle$) does not belong to $\mathcal{H}_{S^1}^\dagger$ (resp. $\mathcal{H}_{S^1}$) but is a formal sum of elements therein. If $\partial\Sigma$ consists of several incoming components S_i^1 and outgoing components S_j^1, we have a map $f_\Sigma : \otimes_i \mathcal{H}_{S_i^1} \to \otimes_j \mathcal{H}_{S_j^1}$. The partition function on Σ with the boundary conditions $\{a_j\}$ and $\{b_i\}$ can be expressed using the boundary states as

$$Z_{\{b_i\}}^{\{a_j\}}(\Sigma) = \Big(\bigotimes_j \langle a_j|\Big) f_\Sigma \Big(\bigotimes_i |b_i\rangle\Big). \tag{3.49}$$

For instance, let us consider a flat finite size cylinder Σ of length T and circumference β. With a choice of orientation in the circle direction, we have one incoming and one outgoing boundaries. We choose the boundary conditions b and a there. Then, the partition function is given by $Z_b^a(\Sigma) = \langle a|\mathrm{e}^{-TH(\beta)}|b\rangle$, where $H(\beta)$ is the Hamiltonian of the theory on the circle of circumference β. This is the interpretation of the partition

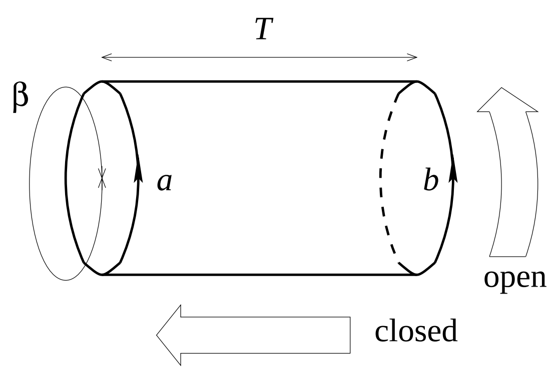

Figure 10: Open and closed string channels

function from the closed string view point. On the other hand, one can interpret it from the point view of open strings. Let $\mathcal{H}_{ab}$ be the space of states on the interval of length T with a and b as the left and the right boundary conditions and let $H(T)$ be the Hamiltonian generating the evolution in the circle direction. If the theory has spin half

fermions and if the spin structure is periodic (anti-periodic) along the circle direction, the partition function is the trace of $(-1)^F e^{-\beta H(T)}$ ($e^{-\beta H(T)}$) over $\mathcal{H}_{ab}$. Thus, we have

$$\mathrm{Tr}_{\mathcal{H}_{ab}}(-1)^F e^{-\beta H(T)} = {}_{\mathrm{RR}}\langle a|e^{-TH(\beta)}|b\rangle_{\mathrm{RR}}, \tag{3.50}$$

$$\mathrm{Tr}_{\mathcal{H}_{ab}} e^{-\beta H(T)} = {}_{\mathrm{NS}^2}\langle a|e^{-TH(\beta)}|b\rangle_{\mathrm{NS}^2}, \tag{3.51}$$

where RR (NS^2) shows that the fermions on the circle are periodic (anti-periodic).

Let us consider a $(2,2)$ supersymmetric field theory formulated on the strip $\Sigma = \mathbf{R}\times I$ of Minkowski signature, with the boundary conditions a and b that preserves A- or B-type supersymmetry. We recall that the x^1 components of the supercurrents are required to obey $\overline{G}^1_+ + G^1_- = G^1_+ + \overline{G}^1_- = 0$ for A-type supersymmetry and $\overline{G}^1_+ + \overline{G}^1_- = G^1_+ + G^1_- = 0$ for B-type supersymmetry (for the trivial phases $e^{i\alpha} = e^{i\beta} = 1$). Now let us compactify the time direction $\mathbf{R}$ to S^1 and continue the theory to Euclidean signature by the Wick rotation $x^0 = -ix^2$ where we choose the orientation so that $z = x^1 + ix^2$ is a complex coordinate. The boundary conditions of the supercharges remains the same as in the Minkowski theory. If we change the coordinates as $(x^{1'}, x^{2'}) = (x^2, -x^1)$, the conditions become $e^{\frac{\pi i}{4}}\overline{G}^{2'}_{+'} + e^{-\frac{\pi i}{4}}G^{2'}_{-'} = e^{\frac{\pi i}{4}}G^{2'}_{+'} + e^{-\frac{\pi i}{4}}\overline{G}^{2'}_{-'} = 0$ for A-type supersymmetry and $e^{\frac{\pi i}{4}}\overline{G}^{2'}_{+'} + e^{-\frac{\pi i}{4}}\overline{G}^{2'}_{-'} = e^{\frac{\pi i}{4}}G^{2'}_{+'} + e^{-\frac{\pi i}{4}}G^{2'}_{-'} = 0$ for B-type supersymmetry, where the phases $e^{\pm\frac{\pi i}{4}}$ come from the spin of the supercurrent. This means that the boundary states satisfy

$$\Big(\overline{G}^{2'}_{+'} - iG^{2'}_{-'}\Big)|b\rangle = \Big(G^{2'}_{+'} - i\overline{G}^{2'}_{-'}\Big)|b\rangle = J^{2'}_A|b\rangle = 0, \tag{3.52}$$

$$\langle a|\Big(\overline{G}^{2'}_{+'} - iG^{2'}_{-'}\Big) = \langle a|\Big(G^{2'}_{+'} - i\overline{G}^{2'}_{-'}\Big) = \langle a|J^{2'}_A = 0 \tag{3.53}$$

for A-type supersymmetry and

$$\Big(\overline{G}^{2'}_{+'} - i\overline{G}^{2'}_{-'}\Big)|b\rangle = \Big(G^{2'}_{+'} - iG^{2'}_{-'}\Big)|b\rangle = J^{2'}_V|b\rangle = 0, \tag{3.54}$$

$$\langle a|(\overline{G}^{2'}_{+'} - i\overline{G}^{2'}_{-'}\Big) = \langle a|\Big(G^{2'}_{+'} - iG^{2'}_{-'}\Big) = \langle a|J^{2'}_V = 0, \tag{3.55}$$

for B-type supersymmetry. Here we have added the conditions for conservation of the R-charge, which applies when the R-symmetry is not broken in the bulk theory. Note that, in the quantization of the closed strings, the Hermiticity condition is imposed so that $(G^{\mu'}_{\pm'})^\dagger = \overline{G}^{\mu'}_{\pm'}$ (whereas the quantization of open strings would lead to $(G^{\mu}_{\pm})^\dagger = \overline{G}^{\mu}_{\pm}$). Thus, the above conditions on the boundary states are not invariant under Hermitian conjugation. If $|b\rangle$ and $\langle a|$ correspond to the boundary conditions preserving A- or B-type supersymmetry with the phase $e^{i\alpha}$ or $e^{i\beta}$, the Hermitian conjugates $\langle\overline{b}|$ and $|\overline{a}\rangle$ correspond to the boundary conditions preserving A- or B-type supersymmetry with the phase $-e^{i\alpha}$ or $-e^{i\beta}$. If the sign flip $(-1)^{F_L}$ of the left-moving worldsheet fermions is a symmetry of

the theory, the states $\langle\overline{b}|(-1)^{F_L}$ and $(-1)^{F_L}|\overline{a}\rangle$ correspond to the boundary conditions preserving the A- or B-type supersymmetry with the phase $\mathrm{e}^{i\alpha}$ or $\mathrm{e}^{i\beta}$, which is the same as the original supersymmetry.[1]

As above, let a and b be the boundary conditions that preserve the same combinations of the supercharges (A-type or B-type). We can use the boundary states to represent the supersymmetric index as

$$I(a,b) = {}_{\mathrm{RR}}\langle a|\mathrm{e}^{-TH(\beta)}|b\rangle_{\mathrm{RR}}, \tag{3.56}$$

where ${}_{\mathrm{RR}}\langle a|$ and $|b\rangle_{\mathrm{RR}}$ are the boundary states in the RR sector. By the basic property of the index, it is independent of the various parameters, such as β and T. It is an integer and therefore must be invariant under the complex conjugation that induces the replacement $(a,b) \to (\overline{b},\overline{a})$. We note, however, that the latter preserves a different combination of the supercharges compared to the original one.

3.3.1 Boundary Entropy

The boundary states are in general a sum of infinitely many eigenstates of the Hamiltonian. An important information on the boundary states can be obtained by looking at the contribution by the ground state. For instance, in boundary conformal field theory, the coefficient $g_b = \langle 0|b\rangle$ of the expansion is known to play a role analogous to that of the central charge c in the bulk theory [34] and is called *boundary entropy*. In supersymmetric field theory, there are several supersymmetric ground states $|i\rangle$ in the RR sector. Thus the $\mathcal{N}=2$ analog of the boundary entropy would be the pairings

$$\Pi_i^a = {}_{\mathrm{RR}}\langle a|i\rangle. \tag{3.57}$$

These overlaps were studied in [2] especially on the relation to the period integrals, which will be further elaborated here. If the axial R-symmetry is unbroken in the bulk theory, we see from (3.53) that an A-type boundary state $\langle a|$ has zero axial charge. Thus for the pairing (3.57) to be non-vanishing, the ground state $|i\rangle$ must also have zero axial R-charge. Likewise, if vector R-symmetry exists, the pairing (3.57) for B-type boundary state is non-vanishing only for the ground state $|i\rangle$ with zero vector R-charge. If the theory has a mass gap, this selection rule is vacuous since all ground states have zero R-charges. However, if there is a non-empty IR fixed point, some of the ground states can have non-vanishing R-charges and this selection rule is non-trivial. For example, in

[1] Our convention differs from that in the reference [30, 32] where $\langle b|$ stands for the Hermitian conjugate of $|b\rangle$. In particular, we do not need an extra $(-1)^{F_L}$ insertion in the r.h.s. of eq. (3.56) that is required in the notation of [30, 32].

LG models all ground states have vanishing R-charges (even if it is quasihomogeneous and has a non-trivial fixed point) and the selection rule is vacuous for A-type boundary states, but in LG orbifold, there are usually ground states of nonzero axial R-charges and the selection rule is non-trivial.

If the vector (resp. axial) R-charge is conserved and integral, there is a one to one correspondence between the supersymmetric ground states and the elements of the ac ring (resp. cc ring) [33]. The state $|\phi_i\rangle$ corresponding to a chiral ring element ϕ_i is the one that appears at the boundary S^1 of the semi-infinite cigar Σ with the insertion of ϕ_i at the tip, where the theory is twisted to a topological field theory in the curved region. Thus, for those states, the pairings ${}_{\rm RR}\langle a|\phi_i\rangle$ can be identified as the path-integral on the

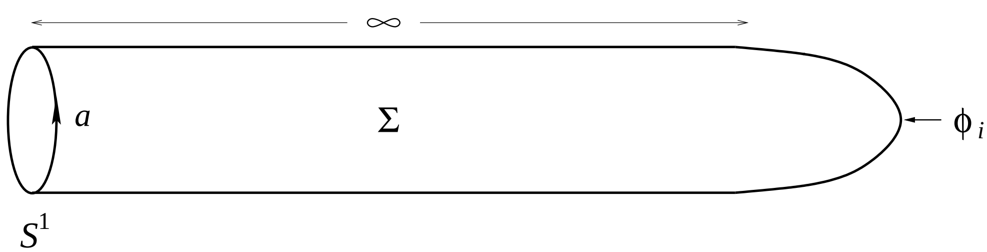

Figure 11: The semi-infinite cigar leading to $\Pi_i^a = {}_{\rm RR}\langle a|\phi_i\rangle$

semi-infinite cigar where the boundary condition a is imposed at the outgoing boundary and the operator ϕ_i is inserted at the tip (see Figure 11).

For concreteness, let us consider a theory with conserved and integral axial R-charge where B-twist is possible. The operators ϕ_i we use to define the supersymmetric ground states are the cc ring elements. We will be interested in A-type boundary conditions a and the corresponding boundary states $\langle a|$ obeying (3.53) which in particular yields

$$\langle a|\Big(\oint_{S^1}\overline{G}_+ - i\oint_{S^1} G_-\Big) = \langle a|\Big(\oint_{S^1} G_+ - i\oint_{S^1}\overline{G}_-\Big) = 0. \tag{3.58}$$

Here we use the closed string coordinates as in (3.53) but omit the primes. Also, we use the current notation $G_\pm = {\rm d}x^1 G_\pm^2 - {\rm d}x^2 G_\pm^1$ which are one forms with values in the spinor bundles of Σ. After B-twisting, the currents $\overline{G}_\pm$ become ordinary one forms but the current G_- (resp. G_+) becomes a one form with values in the bundle of holomorphic (resp. antiholomorphic) one forms of Σ.

The pairings $\Pi_{a,i} = {}_{\rm RR}\langle a|\phi_i\rangle$ are invariant under the twisted F-term deformations of the theory.

$$\frac{\partial \Pi_i^a}{\partial t_{ac}} = 0, \quad \frac{\partial \Pi_i^a}{\partial \bar{t}_{ac}} = 0. \tag{3.59}$$

As for the F-term deformations generated by cc ring elements, they satisfy the following equation

$$(\nabla_i \Pi^a)_j = (D_i \delta_j^k + i\beta C_{ij}^k)\Pi_k^a = 0, \quad (\nabla_{\bar{i}} \Pi^a)_{\bar{j}} = (D_{\bar{i}} \delta_{\bar{j}}^{\bar{k}} - i\beta C_{\bar{i}\bar{j}}^{\bar{k}})\Pi_{\bar{k}}^a = 0, \tag{3.60}$$

where β is the circumference of the boundary circle S^1. Here D_i is the covariant derivative defined in [33] and C_{ij}^k is the structure constant of the chiral ring. These can be shown as follows by the standard gymnastics in tt^* equation.

We start with the twisted F-term deformation which can be written as [2]

$$\frac{1}{2}\int_\Sigma \overline{Q}_- Q_+ \phi_{ac} \sqrt{h} \mathrm{d}^2 x \tag{3.61}$$

plus its complex conjugate. Here ϕ_{ac} is a twisted chiral operator of axial R-charge 2 (and therefore $\overline{Q}_- Q_+ \phi_{ac}$ has spin zero even in the twisted theory; the spin of Q_+ cancels that of ϕ_{ac}). Now, let us divide the semi-infinite cigar into two infinite regions Σ_1 and Σ_2 separated by a circle $S^1_{\rm mid}$ as shown in Figure 12. We first consider the integral (3.61) in

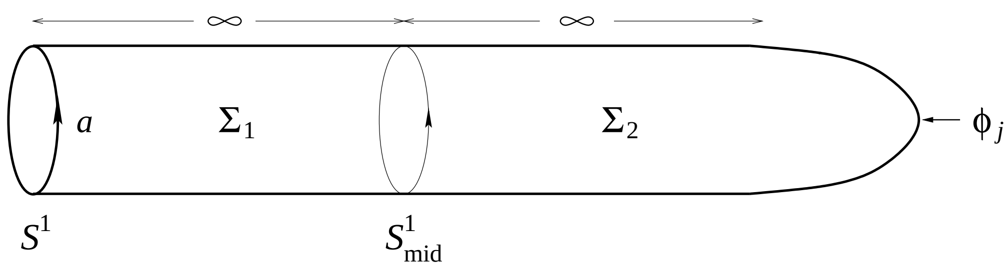

Figure 12: Separation into two regions

the region Σ_1. We recall that $(Q_+\phi_{ac})(x) = \oint_x G_+ \phi_{ac}(x)$ etc, where the contour integral is along a small circle around the point x. We can deform the contour of the $\overline{G}_-$ integral from the small circle around x to two boundary circles of Σ_1; S^1 and $S^1_{\rm mid}$. The one on $S^1_{\rm mid}$ can be considered as the supercharge acting on the state at the boundary of Σ_2. Since Σ_2 is infinitely long, the state that appears at the boundary is a ground state. Thus, the contour integral along $S^1_{\rm mid}$ vanishes. The one on S^1 turns into $i\oint_{S^1} G_+$ by the boundary condition (3.58). By deforming the contour back into the interior, it becomes the sum of the integral on a small circle around x and the one on $S^1_{\rm mid}$. The latter vanishes for the same reason as before. The former becomes $iQ_+Q_+\phi_{ac}$ which is zero from the nilpotency of Q_+. We next consider the region Σ_2. This has a curved region but one can still deform

[2] Note that $(Q_+\phi_{ac})(x) = \oint_x G_+ \phi_{ac}(x)$ where the contour integral is along a small circle around the point x.

the contour of the $\overline{G}_-$ integral since it is an ordinary one-form on Σ. The contour can be deformed to the sum of $S^1_{\rm mid}$ and a small circle around the tip x_0 at which ϕ_j is inserted. The integral on $S^1_{\rm mid}$ can be considered as the supercharge acting on the (dual) state that appears on the incoming boundary of Σ_1. Since Σ_1 is infinite, the state is a ground state and thus the supercharge vanishes. The contour integral around x_0 yields $\overline{Q}_-\phi_j$ but this vanishes since ϕ_j is a chiral operator. To summarize, the pairing $\Pi_{i,a}$ is independent of the twisted F-term deformations.

Next, we consider the F-term deformation generated by the chiral operator ϕ_i, which can be written as

$$\frac{1}{2}\int_\Sigma Q_-Q_+\phi_i + \frac{1}{2}\int_\Sigma \overline{Q}_+\overline{Q}_-\overline{\phi}_{\bar{\imath}}\sqrt{h}\mathrm{d}^2x - i\oint_{S^1}\mathrm{d}x^1(\phi_i - \overline{\phi}_{\bar{\imath}}). \tag{3.62}$$

Here we have included the (constant) boundary term which is required to set the ground state energy at zero, as found before in this section (see (3.30)). We separate the integral into two regions as before. We first consider the region Σ_1. Since Σ_1 is flat, one can treat the currents $G_\pm$ as ordinary one forms. The contour of the G_- integral can be deformed to the one on S^1 and the one on $S^1_{\rm mid}$. The latter vanishes by the same reason as before. By the boundary condition the former turns to $-i\oint_{S^1}\overline{G}_+$ which can be deformed to an integral around the insertion point of ϕ_i (plus an integral over $S^1_{\rm mid}$ that vanishes). This yields the term $-\frac{i}{2}\overline{Q}_+Q_+\phi_i$ which is the same as $-\frac{i}{2}\{\overline{Q}_+, Q_+\}\phi_i$ since ϕ_i is a chiral operator. On the other hand, the same manipulation for G_+ rather than G_- leads to $-\frac{i}{2}\{Q_-, \overline{Q}_-\}\phi_i$. By taking the average of the two we obtain

$$\begin{aligned}\frac{1}{2}\int_{\Sigma_1} Q_-Q_+\phi_i &= \int_{\Sigma_1}\left(-\frac{i}{4}\{\overline{Q}_+, Q_+\}\phi_i - \frac{i}{4}\{Q_-, \overline{Q}_-\}\phi_i\right)\mathrm{d}^2x = -i\int_{\Sigma_1} H\phi_i\,\mathrm{d}^2x \\ &= i\int_{\Sigma_1}\frac{\partial}{\partial x^2}\phi_i\,\mathrm{d}^2x = i\oint_{S^1}\phi_i\,\mathrm{d}x^1 - i\oint_{S^1_{\rm mid}}\phi_i\,\mathrm{d}x^1,\end{aligned} \tag{3.63}$$

where we have used the supersymmetry algebra $\{Q_\pm, \overline{Q}_\pm\} = 2(H \mp P)$. The first term on the right hand side cancels the boundary term in (3.62). The integrand of the last term is a constant along the circle and hence we obtain $-i\beta\phi_i$. One can move the operator ϕ_i toward the tip x_0 where ϕ_j is inserted, and this will yield the term $-i\beta C_{ij}^k\phi_k$. Next we consider the term $\frac{1}{2}\int_{\Sigma_2} Q_-Q_+\phi_i$. Since we have an infinite cylinder to the left of Σ_2, by definition, this yields the term $A_{ij}^k\phi_k$ where $-A_{ij}^k\mathrm{d}t^i$ is the connection form defining the covariant derivative D_i. Thus, we obtain

$$\partial_i\Pi_j^a = A_{ij}^k\Pi_k^a - i\beta C_{ij}^k\Pi_k^a. \tag{3.64}$$

This is nothing but the first equation in (3.60). The derivation of the second equation is similar.

3.3.2 Period Integral as the Boundary Entropy for LG Models

We study the pairings Π_i^a in a LG model on a non-compact Calabi-Yau manifold X, where the axial R-charge is conserved and integral. We consider the D-brane wrapped on a wave-front trajectory γ_a emanating from a critical point in the positive real direction. The corresponding boundary condition a preserves A-type supersymmetry. As we have shown, the pairings $\Pi_i^a = \langle a|\phi_i\rangle$ are invariant under twisted F-term deformations (3.59). In particular they are invariant under the Kähler deformation and can be studied by taking the large volume limit where the contribution of constant maps dominates. We thus expect the quantum mechanical expression $\Pi_i^a = \int_{\gamma_a}\omega_i$ where ω_i are the vacuum wave functions corresponding to the chiral fields ϕ_i. It is known that ω_i are middle dimensional differential forms on X [35] which have the right dimension to be integrated over the middle dimensional cycles γ_a. However, we recall that, in addition to the ordinary path integral with the boundary condition a, we have the following boundary term in the Euclidean action

$$\frac{i}{2}\oint \mathrm{d}x^1\Big(W-\overline{W}\Big) \tag{3.65}$$

which comes from the shift (3.30). This is simply $i\beta(W-\overline{W})/2$ since the integrand is a constant. Thus, the pairing is given by

$$\Pi_i^a = \int\limits_{\gamma_a} \mathrm{e}^{-i\beta(W-\overline{W})/2}\omega_i. \tag{3.66}$$

The vacuum wave forms ω_i are in general difficult to determine. However, a simplification is expected when we make a replacement $(W,\overline{W})\to(\lambda W,\overline{\lambda W})$ and take the limit $\overline{\lambda}\to 0$ but keeping λ finite. In this limit (and in the quantum mechanical approximation), the supercharges $Q_\pm$, $\overline{Q}_\pm$ correspond to the operators

$$\begin{aligned}\overline{Q}_+ &\propto \overline{\partial}-\tfrac{i}{2}\lambda\partial W\wedge, & Q_+ &\propto \overline{\partial}^\dagger,\\ \overline{Q}_- &\propto \partial^\dagger-\tfrac{i}{2}\lambda(\overline{\partial W}\wedge)^\dagger, & Q_- &\propto \partial.\end{aligned} \tag{3.67}$$

The vacuum wave forms ω_i in the $\overline{\lambda}\to 0$ limit must be annihilated by these operators. Reference [35] studies the cohomology of the Dolbeault operator deformed as $\overline{Q}_+$ in (3.67). Under suitable assumption about X[3], it was shown that the cohomology of $\overline{\partial}-\frac{i}{2}\lambda\partial W\wedge$ is isomorphic to the cohomology of Kozsul complex given by the operator $\partial W\wedge$ acting on the holomorphic forms. Furthermore, under the assumption that W has a finite number of critical points, the latter cohomology group is non-zero only at middle dimension and

[3]The assumption is that X be a Stein space, where ordinary Dolbeault cohomology $H^{p,q}(X)$ vanishes except $q=0$ where it is isomorphic to the space of holomorphic p-forms.

is isomorphic to the underlying group of the local ring of W which is nothing but the chiral ring of the LG model. Here we use this fact and the arguments in [35] to study the overlap integral (3.66) in the $\overline{\lambda} \to 0$ limit. The vacuum wave form $\omega = \omega_i$, which in particular defines a $\overline{Q}_+$ cohomology class, can be written as

$$\omega = \Omega + (\overline{\partial} - \frac{i}{2}\lambda \partial W \wedge)\eta, \tag{3.68}$$

where Ω is a holomorphic n-form (where n is the complex dimension of X) and η is an $(n-1)$-form. It is clear that a holomorphic n-form Ω is annihilated by all operators $Q_\pm$, $\overline{Q}_\pm$ in (3.67). Now, let us evaluate the overlap integral

$$\begin{aligned} \lim_{\overline{\lambda} \to 0} \Pi^a &= \int_{\gamma_a} \mathrm{e}^{-i\beta\lambda W/2}\omega = \int_{\gamma_a} \mathrm{e}^{-i\beta\lambda W/2}\Big(\Omega + (\overline{\partial} - \frac{i}{2}\lambda\partial W\wedge)\eta\Big) \\ &= \int_{\gamma_a} \mathrm{e}^{-i\beta\lambda W/2}\Omega + \int_{\gamma_a} \Big\{\mathrm{d}\Big(\mathrm{e}^{-i\beta\lambda W/2}\eta\Big) - \mathrm{e}^{-i\beta\lambda W/2}\partial\eta\Big\}. \end{aligned} \tag{3.69}$$

The total derivative term vanishes under the assumption that the integrand vanishes at infinity of γ_a. Let us focus on the term involving $\partial\eta$. Here we use the fact that the vacuum wave form ω must be annihilated by all supercharges, not just by $\overline{Q}_+$. In particular it must be annihilated by $Q_- \propto \partial$. Since Ω is trivially annihilated by ∂, this leads to the condition $\partial\overline{Q}_+\eta = 0$. Since ∂ and $\overline{Q}_+$ anti-commute with each other, this means that $\partial\eta$ is annihilated by $\overline{Q}_+$. In particular, it can be written as

$$-\partial\eta = \Omega_1 + (\overline{\partial} - \frac{i}{2}\lambda\partial W\wedge)\eta_1, \tag{3.70}$$

where Ω_1 is a holomorphic n-form and η_1 is an $(n-1)$-form. Inserting this expression to (3.69), we obtain $\int_{\gamma_a} \mathrm{e}^{-i\beta\lambda W/2}(\Omega + \Omega_1 - \partial\eta_1)$, where again we assumed that the total derivative term vanishes. Since $\partial\eta$ has no $(0,n)$ component, we can choose η_1 to have no $(0, n-1)$ component. Continuing this procedure, we finally obtain an expression of Π^a as an integral over γ_a of $\mathrm{e}^{-i\beta\lambda W/2}$ times a holomorphic n-form only. The vanishing of the total derivative terms is assured by taking $\lambda = -i$, since the exponential factor becomes $\mathrm{e}^{-\beta W/2}$ which quickly vanishes at infinity of γ_a which extends to real positive directions in the W-plane. Thus, we obtain

$$\lim_{\substack{\lambda \to -i \\ \overline{\lambda} \to 0}} \Pi_i^a = \int_{\gamma_a} \mathrm{e}^{-\beta W/2}\Omega_i. \tag{3.71}$$

where Ω_i is a holomorphic n-form.[4] Even though we considered the limit $\overline{\lambda} \to 0$ in finding

[4] In this derivation we have assumed that X is a Stein space. In the cases of interest in this paper we will be dealing this is the case for some examples. But we will also also consider cases where X has a non-trivial π_1. In such a case one can repeat the arguments above for the covering space and obtain the same results.

this overlap between ground states and D-brane boundary states, in the conformal limit this is unnecessary (as the conformal limit corresponds to taking $\lambda, \overline{\lambda} \to 0$). We will use this result in section 5 in the context of the LG realization of minimal models.

The result obtained here was anticipated in part in [33] and can be viewed as an interpretation of some of the observations there. The argument presented there shows that for flat coordinates, i.e. for a special choice of chiral fields, the period integrals Π_i^a given above satisfy the holomorphic part of the flatness equations given in eq. (3.60). The anti-holomorphic part trivializes in the limit $\overline{\lambda} \to 0$, and thus we obtain the above result in this limit.

4 Brane Creation

As we have already discussed the D-branes preserving the A-type supersymmetry in a LG theory are Lagrangian submanifolds, and their image in the W-plane correspond to straight lines. The slope of the straight lines depend on which phase combination of A-type supercharges one preserves. In particular for $Q_A^\alpha = \overline{Q}_+ + \exp(i\alpha)Q_-$ the image in the W-plane forms an angle α relative to the real axis. Moreover D-branes which lead to boundary states with finite overlap with Ramond ground states correspond to D-branes whose image in the W-plane correspond to straight lines emanating from a critical point.

We consider a LG model of n variables which are coordinates of $\mathbf{C}^n$. Let us assume that the superpotential W has N isolated critical points $x_1, \ldots, x_N$. We denote by B_α the region in $\mathbf{C}^n$ on which $\mathrm{Re}[e^{-i\alpha}W]$ is larger than a fixed large value. Let γ_i be the wavefront trajectory emanating from the critical point x_i along the straight line in the W-plane with the angle α against the real axis. As discussed before γ_i are the cycles on which the D-branes can wrap without breaking the supersymmetry Q_A^α. It is known [14] that the cycles γ_i form a basis of the middle-dimensional homology group $H_n(\mathbf{C}^n, B_\alpha)$ relative to the boundary B_α. In other words $H_n(\mathbf{C}^n, B_\alpha)$ can be viewed as the lattice of charge for the Dn branes.

Now, let us consider a one parameter family $\gamma_1(t)$ of D-branes emanating from a critical point x_1. Here t is a deformation parameter either in the couplings in the superpotential W, or the angle α in the combination of supercharges the D-brane preserves. In such a situation a special thing may happen: The image of the $\gamma_1(t)$ brane in the W-plane may pass through a critical value $W(x_2)$ at some $t = t_0$, so that as we go from $t_0 - \epsilon$ to $t_0 + \epsilon$ the position of the critical value relative to the image of the D-brane on the W-plane, goes from one side to the other. In such a case $\gamma_1(t_0 - \epsilon)$ and $\gamma_1(t_0 + \epsilon)$ will label different

elements of $H_n(\mathbf{C}^n, B_\alpha)$, i.e. they will have different D-brane charges. In particular as discussed before,

$$[\gamma_1(t_0-\epsilon)] = [\gamma_1(t_0+\epsilon)] - (\Delta_1 \circ \Delta_2)\, [\gamma_2(t_0+\epsilon)]\,. \tag{4.1}$$

In the context of string theory, charge conservation would imply that we have to have created $+\Delta_1 \circ \Delta_2$ of γ_2 branes in order to guarantee charge conservation during this process. In the present context the same can be said if we demand *continuity* of the corre-

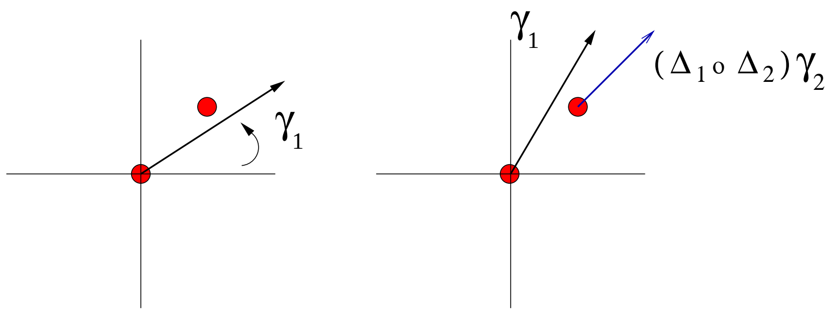

Figure 13: As the image of D-brane in the W-plane passes through a critical value, new D-branes are created whose image start from the crossed critical value.

lation functions of the 2-dimensional theory. For example if we consider the 2-dimensional theory on a cylinder with one boundary ending on the $\gamma_1(t)$, then the continuity of the correlation function with this boundary condition as a function of t demands that as we change t from $t_0-\epsilon$ to $t_0+\epsilon$ we would have to also add to the correlation function the correlator involving boundaries on the γ_2 brane with multiplicity factor $+\Delta_1 \circ \Delta_2$. Also, continuity of the overlap with the Ramond ground states in the closed channel will already imply this. Note that $\Delta_1 \circ \Delta_2$ possibly being negative simply means we have an opposite orientation for the γ_2-brane (i.e. the rules of Grassmann integration over the fermions has picked an extra minus sign).

In string theory a similar process was discovered in [36] where again charge conservation leads to creation of new branes.

4.1 Monodromy and R-Charges

Now we revisit a result obtained in [3] which relates the number of BPS solitons in 2d theories with $(2,2)$ supersymmetry to the R-charges of the Ramond ground states at the conformal point. In particular we show how this result follows very naturally from the

realization of D-branes in LG theories, together with the Brane creation discussed above. Our proof will be based on the case of LG theories, though the generalization to arbitrary massive theories should hold, as already shown in [3].

Consider an LG theory with N isolated massive vacua. As discussed before we can associate N natural D-branes to these vacua, one for each critical point. The image in the W-plane is a straight line emanating from the critical point and going to infinity along a line whose slope depends on the combination of A-type supercharges we are preserving. In particular for $Q_A^\alpha = \overline{Q}_+ + \exp(i\alpha)Q_-$ they make an angle α relative to the real axis.

Let us start with $\alpha = 0$ and order the N D-branes according to the lower value for $Im(W)$, as depicted in Fig. 14. Let us further assume that the critical values have a

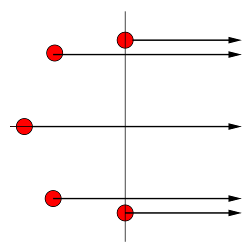

Figure 14: A convex arrangement of critical values in the W-plane is the most convenient one for deriving the monodromy action by 2π rotation.

convexity compatible with the ordering of $Im(W)$ as shown in the figure. This can be done, by deforming the coefficients of W if necessary. As we increase α from 0 to π we rotate the image of branes in the W-plane counter clockwise. As discussed in the previous section, during this process we create new branes. In particular the action of brane creation in the basis of branes emanating from the critical points γ_i is rather simple: The rotation of branes by π in the W-plane causes the γ_i brane to cross all the other γ_j branes with $j > i$ exactly once. Moreover during this crossover it creates $(\Delta_i \circ \Delta_j)$ new γ_j branes. This action of rotation of branes by π is thus realized by an $N \times N$ upper triangular matrix with 1 on the diagonal and $\Delta_i \circ \Delta_j$ for each $i < j$. This is denoted by

$$S = 1 + A\,, \tag{4.2}$$

where A is the upper triangular matrix of inner product of Δ's.

Now consider instead going from $\alpha = 0$ to $\alpha = -\pi$. In this case for each $i > j$ we get $\Delta_i \circ \Delta_j$ brane creation of γ_j. Thus this action is realized as

$$S^t = 1 + A^t\,. \tag{4.3}$$

Now we consider going around from $\alpha = 0$ to $\alpha = 2\pi$. In this case the action on the γ_i brane basis is given by

$$H = SS^{-t}\,, \tag{4.4}$$

where we used here the fact that going from $\alpha = \pi$ to $\alpha = 2\pi$ is the inverse of the action of changing α from 0 to $-\pi$.

Now consider rescaling the superpotential $W \to \lambda W$ as we send $\lambda \to 0$. In this limit we approach a conformal point. For any λ the monodromy operator H we have obtained is the same, because the rescaling of W does not affect the relative location of D-branes or their intersection numbers.

Consider the boundary states $|\gamma_i\rangle$ corresponding to the i-th D-brane at $\alpha = 0$. At the conformal point we obtain a new conserved R-charge, which is the fermion number of the right-moving fermions. In particular we have

$$Q_A^\alpha = \exp(-i\alpha R) Q_A^0 \exp(i\alpha R)\,, \tag{4.5}$$

where R denotes the right moving fermion number charge. Thus the H monodromy is realized in the conformal limit as

$$H|\gamma_i\rangle = \exp(2\pi i R)|\gamma_i\rangle\,. \tag{4.6}$$

On the other hand we can go to a basis where the action of R is diagonal. Note that since the $|\gamma_i\rangle$ have invertible overlaps with the Ramond ground states, we can choose linear combination of Ishibashi type states associated to Ramond ground states to represent $|\gamma_i\rangle$. We thus learn that,

$$\text{Eigenvalues}[SS^{-t}] = \text{Spectrum}[\exp(2\pi i R)] \text{ on Ramond Ground states}\,, \tag{4.7}$$

which is a result of [3] rederived in a purely D-brane language. Note that the choice we have made in the convexity of the critical values is irrelevant for the final result, in that the brane creation was derived precisely based on the continuity of physical correlation functions. The operator $\exp(2\pi i R)$ is a physical observable and the structure of brane creation guarantees that for any distribution of critical values, going around the W plane by 2π will yield the same operator on the γ_i brane states.

In the next section, after we discuss minimal models we show that we can make a slightly stronger statement than just equating the eigenvalues of SS^{-t} with the spectrum of $\exp(2\pi i R)$. Namely we can actually find the change of basis which diagonalizes SS^{-t} by considering the overlap of chiral fields with definite R charges with the corresponding boundary states. That this should be possible is clear, because the chiral fields provide a basis where R acts diagonally.

5 D-Branes in $\mathcal{N}=2$ Minimal Models

In this section, we study D-branes of $\mathcal{N}=2$ minimal models using their realizations as the infra-red fixed points of Landau-Ginzburg models [12, 11]. We will see that the D-branes of the LG models naturally gives rise to the Cardy states of the minimal models and we will be able to study their properties using purely geometric method. In particular, we will find a beautiful geometric realization of the Verlinde ring for $SU(2)$ level k Wess-Zumino-Witten models as well as a simple understanding of the $\tau \to -\frac{1}{\tau}$ modular transformation matrix S_i^j. We first review the construction of the D-branes in the minimal models and then see how they are realized as the D-branes in the LG models.

5.1 Cardy States, Ishibashi States and $N=2$ Minimal models

$\mathcal{N}=2$ minimal models are unitary $(2,2)$ superconformal field theories in two dimensions with central charge $c=\frac{3k}{k+2}$, where k is a positive integer. They can be viewed as an $SU(2)/U(1)$ super-GKO construction at level k. The superconformal primary fields are labeled by three integers (l,m,s) such that

$$\begin{aligned} l &= 0,\cdots,k\,, \\ m &= -(k+1),\cdots,(k+2)\ (\mathrm{mod}\ 2k+4)\ , \\ s &= -1,0,1,2\ (\mathrm{mod}\ 4)\,, \end{aligned} \tag{5.1}$$

with the constraint $l+m+s\equiv 0$ (mod 2) and field identification $(l,m,s)=(k-l,m+k+2,s+2)$. $s=0,2$ in the NS sector and s=±1 in the Ramond sector. The two different values of s denote the GSO parity of various states in the Ramond or NS sector. The conformal weights and the $U(1)$ charges of the primary fields are (mod integer),

$$h^l_{m,s}=\frac{l(l+2)-m^2}{4(k+2)}+\frac{s^2}{8}\,,\quad q^l_{m,s}=\frac{m}{k+2}-\frac{s}{2}\,. \tag{5.2}$$

The $\mathcal{N}=2$ chiral primary states are $(l,l,0)$ in the NS sector. The related Ramond states $(l,l+1,1)$ can be reached by spectral flow. These models can also be described by the IR fixed point of the LG model with a single chiral superfield X with superpotential $W=X^{k+2}$ [12, 11]. The chiral primary fields X^l correspond to the states $(l,l,0)$ and provide a representation of the chiral ring. Note that there are only $k+1$ chiral primary fields (as l ranges from 0 to k), which correspond to the $k+1$ ground states in the Ramond sector. However there are a total of $(k+1)(k+2)$ primary states in the Ramond sector (up to a choice of GSO action $(-1)^F$).

An A-type boundary state satisfies the following boundary conditions,

$$(L_n - \bar{L}_{-n})|B\rangle = 0\,,\ (J_n - \bar{J}_{-n})|B\rangle = 0\,,\ (G_r^{\pm} + i\,\bar{G}_r^{\mp})|B\rangle = 0\,. \tag{5.3}$$

For a rational conformal field theory it was shown in [6] that the boundary states are linear combinations of "Ishibashi states" on which the left and the right generators of the superconformal algebra are linearly related. Ishibashi state corresponding to the primary state (l, m, s) is given by [6]

$$|l, m, s\rangle\!\rangle = \sum_N |l, m, s; N\rangle \otimes U\Omega\overline{|l, m, s; N\rangle}\,. \tag{5.4}$$

Where U is an anti-linear operator acting only on the right moving sector as $U\bar{\mathcal{O}}_n U^{-1} = (-1)^{h_{\mathcal{O}}}\bar{\mathcal{O}}_n$, Ω is the mirror automorphism of the $\mathcal{N} = 2$ algebra and the states $|l, m, s; N\rangle$ form an orthonormal basis of $\mathcal{H}_{l,m,s}$. The boundary states are particular linear combination of Ishibashi states [7]

$$|l, m, s\rangle_{BS} = \alpha \sum_{(l',m',s')} \frac{S_{l,m,s}^{\ l',m',s'}}{\sqrt{S_{0,0,0}^{\ l',m',s'}}}|l', m', s'\rangle\!\rangle\,. \tag{5.5}$$

Where the constant, α, is fixed by the condition that the partition function in the open sting channel is integral linear combination of the characters. The summation in the above equation is only over allowed states modulo the field identification and S^i_j is the matrix representation of the modular transformation $\tau \mapsto -\frac{1}{\tau}$ for the characters $\chi_{l,m,s}(\tau) = \mathrm{Tr}_{\mathcal{H}_{l,m,s}} q^{L_0 - \frac{c}{24}}$,

$$\chi_{l,m,s}(-\tfrac{1}{\tau}) = \textstyle\sum_{(l',m',s')} S_{l,m,s}^{\ l',m',s'}\chi_{(l',m',s')}(\tau)\,, \tag{5.6}$$

and is given by,

$$S_{l,m,s}^{\ l',m',s'} = \tfrac{1}{\sqrt{2}\,(k+2)}\ \mathrm{Sin}(\pi\tfrac{(l+1)(l'+1)}{k+2})e^{\frac{i\pi mm'}{k+2}}e^{-\frac{i\pi ss'}{2}}\,. \tag{5.7}$$

The above identification of boundary state is motivated mainly by demanding integral expansion in the characters of $Tr_{\alpha,\beta}q^{L_0}$ corresponding to open strings ending on α and β D-branes. The integrality of characters in this sector follows from properties of the Verlinde algebra. We are interested in the Ramond part of the boundary state which can be obtained by restricting the sum in eq. (5.5) to Ramond states only. The properly normalized Ramond part of the boundary state is,

$$|l, m, s\rangle_{\mathrm{RR}} = \sqrt{2\sqrt{2}} \sum_{(l',m',s')_R} \frac{S_{l,m,s}^{\ l',m',s'}}{\sqrt{S_{0,0,0}^{\ l',m',s'}}}|l', m', s'\rangle\!\rangle\,. \tag{5.8}$$

Consider an open string in the (a, b) sector. As we have discussed in section 3, the index $I(a, b) = \mathrm{Tr}_{a,b}(-1)^F \mathrm{e}^{-\beta H}$ corresponds in the closed string channel to an overlap in the

Ramond sector boundary states $I(a,b) = \mathrm{Tr}_{a,b}(-1)^F = {}_{\rm RR}\langle a|b\rangle_{\rm RR}$. Using the expression (5.8), it is straightforward to compute the index in the $(a,b) = ((l_1,m_1,s_1),(l_2,m_2,s_2))$ sector. Since the index gets contribution from the Ramond ground states only we have,

$$\begin{aligned}
I(a,b) &= {}_{\rm RR}\langle l_1,m_1,s_1|l_2,m_2,s_2\rangle_{\rm RR} \\
&= 2\sqrt{2}\sum_{l=0}^{k}\frac{(S^{l,l+1,1}_{l_1,m_1,s_1})^*\,S^{l,l+1,1}_{l_2,m_2,s_2}}{S^{l,l+1,1}_{0,0,0}}\,\langle l,l+1,1|l,l+1,1\rangle \\
&= \tfrac{-2\sqrt{2}i}{\sqrt{2}(k+2)}\sum_{l=0}^{k}\frac{\mathrm{Sin}(\pi\frac{(l+1)(l_1+1)}{k+2})\mathrm{Sin}(\pi\frac{(l+1)(l_2+1)}{k+2})}{\mathrm{Sin}(\pi\frac{l+1}{k+2})}e^{\frac{i\pi(l+1)(m_2-m_1+1)}{k+2}}\,e^{-\frac{i\pi(s_2-s_1)}{k+2}} \\
&= \tfrac{2\,e^{-\frac{i\pi(s_2-s_1)}{2}}}{k+2}\sum_{l=0}^{k}\frac{\mathrm{Sin}(\pi\frac{(l+1)(l_1+1)}{k+2})\mathrm{Sin}(\pi\frac{(l+1)(l_2+1)}{k+2})\mathrm{Sin}(\pi\frac{(l+1)(m_2-m_1+1)}{k+2})}{\mathrm{Sin}(\pi\frac{l+1}{k+2})}\,.
\end{aligned}$$

Finally using the fact that s_2-s_1 is an even integer in the Ramond sector we get [31, 32, 10],

$$I((l_1,m_1,s_1),(l_2,m_2,s_2)) = (-1)^{\frac{s_2-s_1}{2}}\;N_{l_1,l_2}^{\;m_2-m_1}\,. \tag{5.9}$$

Where

$$N^{l_1}_{l_2,l_3} = \tfrac{2}{k+2}\sum_{l=0}^{k}\frac{\mathrm{Sin}(\pi\frac{(l+1)(l_1+1)}{k+2})\mathrm{Sin}(\pi\frac{(l+1)(l_2+1)}{k+2})\mathrm{Sin}(\pi\frac{(l+1)(l_3+1)}{k+2})}{\mathrm{Sin}(\pi\frac{l+1}{k+2})}$$

are the $SU(2)_k$ fusion coefficients,

$$N_{l_1,l_2}^{\;m_2-m_1} = \begin{cases} 1 & \text{if} \quad |l_1-l_2| \le m_2-m_1 \le \min\{l_1+l_2, 2k-l_1-l_2\}\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{5.10}$$

As we have studied in section 3, the index for a pair of D-branes in the LG model can be identified as the "intersection number" of the corresponding cycles. In the LG realization of the minimal model D-branes, as we will see below, the index (5.9) can indeed be considered as such an "intersection number".

5.2 D-branes in LG description

The Landau Ginzburg description of A_{k+1} minimal model consists of a single chiral superfield X with superpotential $W = X^{k+2}$ [12, 11]. In section 3 we saw that the D-branes in the LG description are preimages of the straight lines in W-plane starting from the critical values which correspond to vanishing cycles in the x-space fibered over the W-plane.

The superpotential $W = X^{k+2}$ has a single critical point $X = 0$ of multiplicity $k+1$ with critical value $w_* = 0$. If we consider deforming the superpotential by lower powers

of X we will generically obtain $k+1$ isolated and non-degenerate critical points with distinct critical values w_i. We assume that $\mathrm{Im}(w_i)$ are separate from one another. Then as we discussed before we would get $k+1$ D-branes, one associated to each of the critical points. Moreover the image of these D-branes are straight lines in the W-plane going to $+\infty$ in the real positive direction. For large values of X the lower order terms which deform W are irrelevant and the D-branes approach the preimages of the positive real axis $X^{k+2} \in \mathbf{R}_{\geq 0} \subset \mathbf{C}$, namely

$$X = r \cdot \exp\left(\frac{2\pi n i}{k+2}\right), \quad n = 0, \cdots, k+1, \quad r \in [0, \infty). \tag{5.11}$$

Thus we see that the X-plane is divided up into $k+2$ wedge shaped regions by the $k+2$ lines going from the origin to infinity making an angle of $\frac{2\pi n}{k+2}$ with the positive real axis, we will denote such a line by $\mathcal{L}_n$.

Any D-brane of the deformed theory is a curve in the X-plane that will asymptote to a pair of such lines, say $\mathcal{L}_{n_1}$ and $\mathcal{L}_{n_2}$ with $n_1 \neq n_2$. To see this, we note that the deformed superpotential W is approximately quadratic around any (non-degenerate) critical point a and the preimage of the straight line emanating from $W(a)$ in the W-plane splits to trajectories of two points (wavefronts) starting from a. The two wavefronts approach the lines $\mathcal{L}_{n_1}$ and $\mathcal{L}_{n_2}$ as they move away from the critical point. To see that $n_1 \neq n_2$ it is sufficient to note that the two wavefronts can merge only at a critical point (but the $(k+1)$ critical values are assumed to be separate in the imaginary direction). Note that the homology class of the D-branes is completely specified by the choices of the combinations of $k+2$ wedges in the x-plane, and that the $k+1$ D-branes will be enough to provide a linear basis for the non-trivial cycles (since the sum of all wedges is homologically a trivial contractible cycle). Precisely which combination of $k+1$ pairs of asymptote we obtain will depend on which deformation of W away from criticality we are considering.

In general the $k+1$ D-branes we obtain in this way will not intersect with each other (as their images in the W-plane do not intersect one another). Nevertheless, as discussed in section 3 the index $I(a,b) = \mathrm{Tr}_{a,b}(-1)^F$ is not in general zero and will depend on the number of solutions to (3.31), that is, how many orthogonal gradient trajectories there are from a to b D-branes, with a fixed length $x^1 \in [0, \pi]$. This in turn is given by the "intersection number" of the D-branes which is defined as the geometric intersection number where the b-brane is tilted with a small positive angle in the W-plane.

We thus see that away from the conformal point there are $k+1$ distinct pairs of D-branes, each labeled by an ordered pair of integers (n_1, n_2) which label the asymptotes that it makes (taking into account the orientation of the D-brane). In particular n_1 and n_2 are well defined modulo $k+2$ and $n_1 \neq n_2$. We will label such a D-brane by $\gamma_{n_1 n_2}$.

However there are only $k+1$ such pairs for a generically deformed W. In particular we do not have both branes of the type $\gamma_{n_1n_2}$ and $\gamma_{r_1r_2}$ with $n_1 < r_1 < n_2 < r_2$ for generically deformed W as that would have required them to geometrically intersect.

Let us consider two allowed branes $\gamma_{n_1n_2}$ and $\gamma_{r_1r_2}$. We are interested in computing the Witten index in the oriented open string sector starting from the $\gamma_{n_1n_2}$ brane and ending on the $\gamma_{r_1r_2}$ brane. Let us denote this by the overlap of the corresponding boundary states, namely $\langle\gamma_{n_1n_2}|\gamma_{r_1r_2}\rangle$. If none of the n_i and r_i are equal the branes do not intersect even when one of them is slightly tilted (as noted before in the massive theory the case $n_1 < r_1 < n_2 < r_2$ is not allowed) and thus the index is zero. The more subtle case is when one of the n_i is equal to one of the r_i. If they are both equal then we get the Witten index to be 1 as discussed before. Without loss of generality we can order the branes so that $n_1 < n_2$ and $r_1 < r_2$ (otherwise the intersection number gets multiplied by a minus sign for each switch of order). Thus there are only four more cases to discuss: $n_i = r_j$, for some choice of pair of $i, j = 1, 2$. Let us also assume that $r_1 + r_2 > n_1 + n_2$ (by $r_2 = n_1$ we mean equality mod $k+2$, i.e. this is $r_2 = n_1 + k + 2$). It turns out that in such cases

$$\langle\gamma_{n_1n_2}|\gamma_{r_1r_2}\rangle = 1 \quad \text{iff} \quad n_i = r_i \ \text{ for some } \ i \tag{5.12}$$

and zero otherwise. To see this, as discussed in section 2 and 3 it suffices to consider tilting the slope of the image in the W-plane of the D-brane corresponding to $\gamma_{r_1r_2}$ in the positive direction and seeing if they intersect or not. Tilting the slope in the W-plane in this case will also correspond to tilting the asymptotes r_1, r_2 in the positive direction and seeing if they intersect the n_1, n_2 brane. This leads to the above formula. The case of $r_1 = n_1$ which leads to intersection number 1 and $r_1 = n_2$ which leads to intersection number zero is depicted in the Fig. 15.

Now we come to the D-branes at the conformal point. Since the $k+1$ D-branes make sense arbitrary close to the conformal point, they survive in the limit of conformal point as well. But here since we have different allowed D-branes at the massive theory, depending on the choice of the deformation polynomials, we learn that *all* of them survive at the conformal point. Since all pairs (n_1, n_2) are realized in terms of a D-brane for *some* deformation of W (which follows from Picard-Lefshetz action discussed before) we learn that all D-branes $\gamma_{n_1n_2}$ exist for arbitrary unequal integers n_1, n_2 defined mod $k+2$, which now correspond to exact straight lines in the x space along the half-lines given by n_1 and n_2, passing through the origin. This gives us a total of $(k+2)(k+1)$ D-branes, which are pairwise the same upto orientation at the conformal point. Here we are encountering an interesting effect: *The number of D-branes jump as we go from the conformal point to the massive theory.*

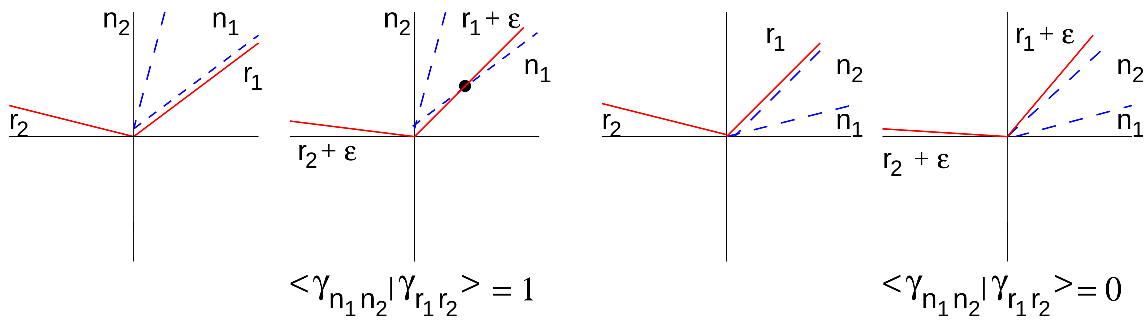

Figure 15: To get the index in the open string sector stretched between $\gamma_{n_1n_2}$ and $\gamma_{r_1r_2}$ we have to rotate the image in the W-plane of the $\gamma_{r_1r_2}$ brane in the positive direction and compute the corresponding intersection number.

The fact that we have obtained $(k+2)(k+1)$ of such branes at the conformal point is very encouraging as that is exactly the same as the predicted number of Cardy states, as already discussed. Moreover, if we consider the range of parameters where $0 \leq n_1 < n_2 \leq k+1$ we see that $|n_1 - n_2| \in \{1, \cdots, k+1\}$ and $n_1 + n_2 \in \{0, \cdots 2k+2\}$. The range of these parameters exactly corresponds to the quantum numbers (l, m) labelling the boundary states. Note also that $s = \pm 1$ for the Ramond sector boundary states which we are considering. Thus we claim the following identification

$$|n_2 - n_1| = l + 1, \qquad n_1 + n_2 = m, \qquad s = \text{sign}(n_2 - n_1). \tag{5.13}$$

It follows from (5.13) that $m + l + s = 2n_1 = 0 \,(\text{mod}\, 2)$ as is required. The field identification $(l, m, s) = (k - l, m + k + 2, s + 2)$ also has a natural identification as shown in Fig. 16 and relates to the statement that if we change $n_1 \to n_2$ and $n_2 \to n_1 + k + 2$ we get the same brane back up to a flip in the orientation (reflected in the shift in s).

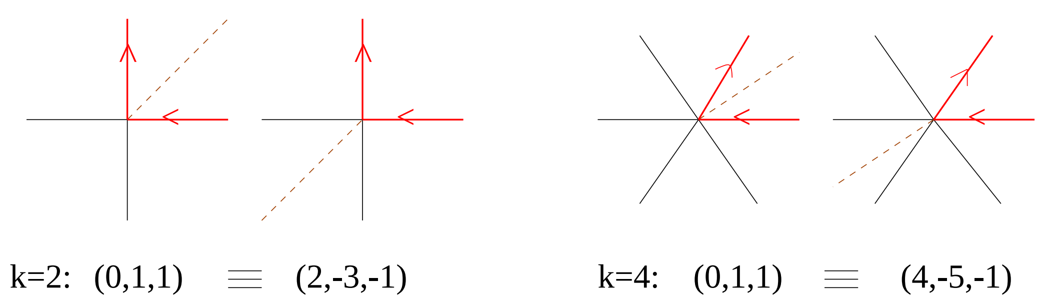

Figure 16: Field identification

It is convenient to choose $n_2 > n_1$ (with their differences less than $k+2$) in which case we have

$$n_2 - n_1 = l + 1, \quad n_2 + n_1 = m, \quad s = 1, \tag{5.14}$$

which can be solved for $2n_1$ and $2n_2$ as

$$2n_1 = m - l - 1, \qquad 2n_2 = m + l + 1. \tag{5.15}$$

We will denote the D-brane corresponding to the Ramond boundary state $|l, m, s\rangle_{RR}$ by $\gamma_{\,l,m,s}$. We will now provide further evidence for this identification. Along the way we find a simple geometric interpretation of Verlinde ring for $SU(2)$ level k, as well as certain matrix elements of modular transformations matrix S.

5.3 Geometric Interpretation of Verlinde Algebra

We would like to compute the Witten index at the conformal point for the open string strechted between two D-branes $\gamma_{n_1 n_2}$ and $\gamma_{r_1 r_2}$ and reproduce the index formula (5.9). One aspect of the formula is clear. The "intersection number" will not change if we rotate both branes by integral multiples of $\frac{2\pi}{k+2}$ which implies that the index will depend on $m_2 - m_1$ but not on the other combination of m_1 and m_2 (as m_1 and m_2 shift by the same amount under the rotation). Moreover the appearance of $(-1)^{\frac{s_2 - s_1}{2}}$ in the intersection is also natural as that correlates with the choice of orientation on the D-branes. So without loss of generality we set $s_1 = s_2 = 1$, i.e., as before we choose $n_2 > n_1$ and $r_2 > r_1$. Also in checking (5.9) in computing the Verlinde algebra coefficients it suffices to consider the case where $m_2 - m_1 \geq 0$ which is the same case as $r_1 + r_2 \geq n_1 + n_2$. With these set, it is now clear what the conditions are for obtaining overlap 1, namely we must have

$$n_1 \leq r_1 < n_2 \leq r_2 < n_1 + k + 2 \tag{5.16}$$

and all the other cases vanish. This is simply the condition that the branes intersect as shown in Fig. 17.

Note that the condition of getting non-vanishing results in the case of equality follows from equation (5.12). Now we use (5.15) to rewrite (5.16) as

$$m_1 - l_1 - 1 \leq m_2 - l_2 - 1 < m_1 + l_1 + 1 \leq m_2 + l_2 + 1 < m_1 - l_1 + 2k + 3 \tag{5.17}$$

These four conditions can also be written as

$$|l_2 - l_1| \leq m \leq \min[l_1 + l_2, 2k - l_1 - l_2] \tag{5.18}$$

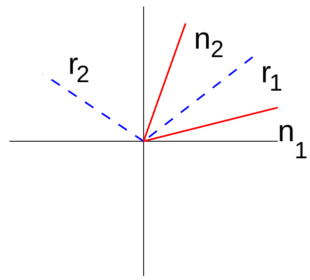

Figure 17: The intersection number of the branes is 1 if the corresponding asymptotes alternate. Otherwise it is zero.

where $m = m_2 - m_1$ (to show this and write all inequalities in terms of inequalities with equal signs we used the fact that $m_2 - m_1$ and $l_2 + l_1$ are equal mod 2). This is precisely the condition for the $SU(2)$ level k algebra and we have thus derived (5.9) from a purely LG point of view.

5.4 Period Integrals and Boundary States

As was discussed in the context of LG models [2] and also in section 3 of this paper there is a natural pairing between the A-model boundary states and B-model chiral fields given by integrating the B-model chiral fields over the cycles representing the A-type boundary states. This kind of pairing was first noticed in [39] and elaborated further in [3]. For the A_{k+1} minimal models the chiral primary fields are X^l and therefore the inner product of the boundary state $|l, m, s\rangle_{\rm RR}$ and the state defined by the B-model chiral field is as discussed in section 3,

$$
{}_{\rm RR}\langle l, m, s|X^{l'}\rangle \;=\; \int_{\gamma_{l,m,s}} dX\; X^{l'} e^{-W(X)}\,,
$$

where the superpotential $W(X) = X^{k+2}$. The image of the cycle $\gamma_{l,m,s}$ in the W-plane is the positive real axis. Thus from the discussion of the previous section we can see that we can parameterize the curve $\gamma_{l,m,1}$ in the following way,

$$
\begin{aligned}
\gamma_{l,m,1} : X(t) &= (-t)^{\frac{1}{k+2}} e^{\frac{i\pi(m-l-1)}{k+2}}, \quad t \in [-\infty, 0]\,, \\
&= t^{\frac{1}{k+2}} e^{\frac{i\pi(m+l+1)}{k+2}}, \qquad t \in [0, \infty]\,.
\end{aligned} \tag{5.19}
$$

Since $m + l + 1 \equiv 0 \pmod 2$, the image in the W-plane of $\gamma_{l,m,1}$ is the positive real axis. We have given the curve for $s = 1$ the curve for $s = -1$ can be obtained from this by

reversing the orientation. With the above parameterization of $\gamma_{l,m,1}$,

$$\begin{aligned}
\int_{\gamma_{l,m,1}} dX X^{l'} e^{-W(X)} &= \frac{e^{\frac{i\pi(m-l-1)(l'+1)}{k+2}}}{k+2}\int_{+\infty}^{0} t^{\frac{l'+1}{k+2}-1}\, e^{-t}\, dt + \frac{e^{\frac{i\pi(m+l+1)(l'+1)}{k+2}}}{k+2}\int_{0}^{+\infty} t^{\frac{l'+1}{k+2}-1}\, e^{-t}\, dt \\
&= \Big(-\frac{e^{\frac{i\pi(m-l-1)(l'+1)}{k+2}}}{k+2} + \frac{e^{\frac{i\pi(m+l+1)(l'+1)}{k+2}}}{k+2}\Big)\int_{0}^{+\infty} t^{\frac{l'+1}{k+2}-1}\, e^{-t}\, dt \\
&= \frac{e^{\frac{i\pi m(l'+1)}{k+2}}}{k+2}(2i\mathrm{Sin}(\pi\tfrac{(l+1)(l'+1)}{k+2})\Gamma(\tfrac{l'+1}{k+2})\,. \qquad (5.20)
\end{aligned}$$

To relate the above integral with the modular S-matrix we need to use the normalized operator $X^{l'}_{norm}$ ([40] [3]) in the above integral. The normalization can be determined by evaluating the matrix element $\langle \bar{X}^{l'}|X^{l'}\rangle$.

To determine the matrix element $\langle \bar{X}^{l'}|X^{l'}\rangle$ note that $\sum_{a_1,a_2}|a_1\rangle_{\rm RR}\,{}_{\rm RR}\langle a_2|S^{a_1a_2} = 1$ when it is sandwitched by the ground states. Here $|a_1\rangle_{\rm RR}$ form a basis of the Ramond boundary states and $S^{a_1a_2}$ is the inverse of the index matrix,

$$S^{a_1a_2} = (S^{-1})_{a_1a_2}\,, \quad S_{a_1a_2} = I(a_1,a_2) = {}_{\rm RR}\langle a_1|a_2\rangle_{\rm RR}\,. \qquad (5.21)$$

We choose the basis such that the state $|a\rangle_{\rm RR}$ corresponds to the D-brane $\mathcal{L}_{a+1} - \mathcal{L}_a$, where $a = 0,\cdots,k$. With this choice of the basis states we see that [1]

$$S_{a_1a_2} = \delta_{a_1,a_2} - \delta_{a_1+1,a_2}\,, \quad S^{a_1a_2} := (S^{-1})_{a_1a_2} = \begin{cases} 1 & , \; a_2 \geq a_1 \\ 0 & , \; a_2 < a_1\,. \end{cases}\,, \qquad (5.22)$$

which follows from eq. (5.12) by taking into account the reversal of the orientation of the neighboring branes. We insert this complete set of states in the matrix element $\langle \bar{X}^{l'}|X^{l'}\rangle$,

$$\langle \bar{X}^{l'}|X^{l'}\rangle = \sum_{a_1,a_2=0}^{k}\langle \bar{X}^{l'}\,|\,a_1\rangle_{\rm RR}\; S^{a_1a_2}\;{}_{\rm RR}\langle a_2\,|\,X^{l'}\rangle\,. \qquad (5.23)$$

Using eq. (5.20) we see that

$$\begin{aligned}
{}_{\rm RR}\langle a_2\,|\,X^{l'}\rangle &= \int_{\mathcal{L}_{a_2+1}} dX\, X^{l'} e^{-W(X)} - \int_{\mathcal{L}_{a_2}} dX\, X^{l'} e^{-W(X)} \\
&= \tfrac{1}{k+2}\{e^{\frac{2\pi i(a_2+1)(l'+1)}{k+2}} - e^{\frac{2\pi i a_2(l'+1)}{k+2}}\}\Gamma(\tfrac{l'+1}{k+2})\,. \qquad (5.24)
\end{aligned}$$

To calculate $\langle \bar{X}^{l'}|a_1\rangle_{\rm RR}$ we use the fact that, as discussed in section 3,

$$\langle \bar{X}^{l'}|a_1\rangle_{\rm RR} = {}_{\rm RR}\langle a_1|(-1)^{F_L}|X^{l'}\rangle^*\,. \qquad (5.25)$$

[1] $S_{a_1a_2}$ is the intersection matrix not to be confused with the modular transformation matrix $S^{l',m',s'}_{l,m,s}$ for which the indices will always be written as subscript and superscript.

Using the action

$$(-1)^{F_L}|X^{l'}\rangle = e^{i\pi(-\frac{\widehat{c}}{2}-\frac{l'}{k+2})}|X^{l'}\rangle = i\, e^{-\frac{i\pi(l'+1)}{k+2}}|X^{l'}\rangle$$

where $\widehat{c} = \frac{c}{3} = \frac{k}{k+2}$ we thus obtain

$$\langle \bar{X}^{l'}|a_1\rangle_{\rm RR} = \frac{-i\, e^{\frac{i\pi(l'+1)}{k+2}}}{k+2}\{e^{-\frac{2\pi i(a_1+1)(l'+1)}{k+2}} - e^{-\frac{2\pi i a_1(l'+1)}{k+2}}\}\Gamma(\tfrac{l'+1}{k+2})\,. \tag{5.26}$$

Using eq. (5.24) and eq. (5.26) in eq. (5.23) we see that

$$\begin{aligned}
\langle \bar{X}^{l'}|X^{l'}\rangle &= -4i\frac{e^{\frac{i\pi(l'+1)}{k+2}}}{(k+2)^2}\,\mathrm{Sin}^2(\pi\tfrac{l'+1}{k+2})\,\Gamma(\tfrac{l'+1}{k+2})^2\textstyle\sum_{a_1,a_2=0}^{k} S^{a_1a_2}e^{\frac{2\pi i(a_2-a_1)(l'+1)}{k+2}}\\
&= -4i\frac{e^{\frac{i\pi(l'+1)}{k+2}}}{(k+2)^2}\,\mathrm{Sin}^2(\pi\tfrac{l'+1}{k+2})\,\Gamma(\tfrac{l'+1}{k+2})^2\textstyle\sum_{a_1=0}^{k}\sum_{a_2=a_1}^{k} e^{\frac{2\pi i(a_2-a_1)(l'+1)}{k+2}}\\
&= \tfrac{2}{k+2}\,\mathrm{Sin}(\pi\tfrac{l'+1}{k+2})\;\Gamma(\tfrac{l'+1}{k+2})^2\,.
\end{aligned}\tag{5.27}$$

Thus we see that

$$X^{l'}_{norm} = \sqrt{\frac{k+2}{2\mathrm{Sin}(\frac{\pi(l'+1)}{k+2})}}\;\frac{X^{l'}}{\Gamma(\frac{l'+1}{k+2})}\,. \tag{5.28}$$

Using the normalized operator $X^{l'}_{norm}$ in eq. (5.20) we get

$$\int_{\gamma_{l,m,s}} dX X^{l'}_{norm}e^{-W(X)} = \sqrt{\frac{2}{(k+2)\mathrm{Sin}(\pi\frac{l'+1}{k+2})}}\; e^{i\pi s/2}\, e^{i\pi\frac{m(l'+1)}{k+2}}\,\mathrm{Sin}(\pi\tfrac{(l+1)(l'+1)}{k+2})\,. \tag{5.29}$$

We can immediately recognize the r.h.s in eq. (5.29) as the coefficient of the Ishibashi state in the expansion of the boundary state i.e.,

$${}_{\rm RR}\langle l,m,s|X^{l'}\rangle = \sqrt{2\sqrt{2}}\;\frac{S_{l,m,s}^{\;l',l'+1,-1}}{\sqrt{S_{0,0,0}^{\;l',l'+1,-1}}} \tag{5.30}$$

Thus we have found a beautiful realization of the modular transformation matrix in terms of classical integrals in the LG theory.[2]

We can actually check more. Namely we know that the Ramond states corresponding to chiral fields $|X^{l'}\rangle$ provides a basis where the R charge is diagonal. This implies that if we consider a basis for the D-branes, for example the one given above, $\gamma_{n,n+1} := \mathcal{L}_{n+1}-\mathcal{L}_n$ where $n = 0,...,k$ and compute the operator SS^{-t} where S is intersection matrix given in eq. (5.22), then the corresponding change of basis to make it diagonal should be given by the matrix

$$M_{ab} := {}_{\rm RR}\langle\gamma_{a,a+1}|X^b\rangle \tag{5.31}$$

[2]Computation of boundary entropy in terms of kinks was carried out in [41] in a slightly different context where the modular S-matrix for $SU(1)_k$ appeared in a similar way. It would be interesting to see whether and how it is related to the present discussion.

where $a, b \in \{0, \cdots, k\}$. To show this we will calculate $D := M^{-1} S S^{-t} M$ and show that it is a diagonal matrix with eigenvalues equal to the R charge.

From eq. (5.29) and eq. (5.30) it follows that the matrix M and its inverse is given by,

$$\begin{aligned} M_{ab} &= -i\sqrt{\tfrac{2}{k+2}}\; e^{\frac{i\pi(2a+1)(b+1)}{k+2}} \sqrt{\mathrm{Sin}(\pi\tfrac{b+1}{k+2})}\,, \\ (M^{-1})_{ab} &= \sqrt{\tfrac{2}{k+2}} \sum_{c=0}^{k} S^{cb}\; e^{-\frac{2\pi i c(a+1)}{k+2}} \sqrt{\mathrm{Sin}(\pi\tfrac{a+1}{k+2})}\,. \end{aligned} \tag{5.32}$$

Now consider D_{ab},

$$\begin{aligned} D_{ab} &= \sum_{c,d,g=0}^{k} (M^{-1})_{ac} S_{cd} (S^{-t})_{dg} M_{gb} \\ &= \tfrac{-2i}{k+2}\sqrt{\mathrm{Sin}(\pi\tfrac{a+1}{k+2})\,\mathrm{Sin}(\pi\tfrac{b+1}{k+2})}\, \textstyle\sum_{g,f=0}^{k} e^{-\frac{2\pi i f(a+1)}{k+2}}\, (S^{-t})_{fg}\, e^{\frac{i\pi(2g+1)(b+1)}{k+2}}\,, \\ &= \tfrac{-2i}{k+2}\sqrt{\mathrm{Sin}(\pi\tfrac{a+1}{k+2})\,\mathrm{Sin}(\pi\tfrac{b+1}{k+2})}\, \textstyle\sum_{g=0}^{k}\sum_{f\geq g}^{k} e^{-\frac{2\pi i f(a+1)}{k+2}}\; e^{\frac{i\pi(2g+1)(b+1)}{k+2}}\,, \\ &= \tfrac{-2i}{k+2}\sqrt{\mathrm{Sin}(\pi\tfrac{a+1}{k+2})\,\mathrm{Sin}(\pi\tfrac{b+1}{k+2})}\; e^{\frac{i\pi(b+1)}{k+2}}\; \textstyle\sum_{g=0}^{k}\sum_{f\geq g}^{k} e^{-\frac{2\pi i f(a+1)}{k+2}}\; e^{\frac{2\pi i g(b+1)}{k+2}}\,. \end{aligned} \tag{5.33}$$

Using the identity

$$\sum_{e=0}^{k}\sum_{f\geq e}^{k} e^{-\frac{2\pi i f(a+1)}{k+2}}\; e^{\frac{2\pi i e(b+1)}{k+2}} = \delta_{a,b}\frac{(k+2)e^{\frac{i\pi(b+1)}{k+2}}}{2\,i\,\mathrm{Sin}(\pi\frac{b+1}{k+2})} \tag{5.34}$$

we see that

$$D_{ab} = -\,e^{\frac{2\pi i(b+1)}{k+2}}\;\delta_{a,b}\,, \tag{5.35}$$

which is indeed the spectrum of $\exp(2\pi i R)$ for the $\mathcal{N}=2$ minimal model. This is morally the analog of the fact that in rational conformal field theory the modular transformation matrix corresponding to $\tau \to -\frac{1}{\tau}$ diagonalizes the fusion algebra N_{ij}^{k} [38]. Namely in this case the intersection matrix S is related to the N_{ij}^{k} coefficients, as already shown, and the M is given by the overlap of Ishibashi states with chiral fields eq. (5.30) which is given in terms of the modular transformation matrix of the rational conformal theory.

6 Boundary Linear Sigma Models

In this section, we study $(2,2)$ supersymmetric gauge theories formulated on a worldsheet with boundary. We seek for boundary conditions that preserve B-type supersymmetry and study its relation to non-linear sigma model to which the theory reduces at low energies. We also analyze how these boundary conditions are described in the dual description that was found in [2]. We include a brief review of the analysis of [2].

6.1 Supersymmetric Boundary Conditions

Let us consider a supersymmetric $U(1)$ gauge theory with chiral multiplets $\Phi_1, \ldots, \Phi_N$ of charge $Q_1, \ldots, Q_N$. We formulate the theory on the strip $\Sigma = \mathbf{R} \times I$ where I is a finite interval parametrized by $x^1 \in [0, \pi]$ and $\mathbf{R}$ is parametrized by the time coordinate x^0. The action of the system is given by

$$S = \frac{1}{2\pi} \int_{\Sigma} \Big(L_{kin} + L_{gauge} + L_{FI,\theta} \Big) \mathrm{d}^2 x. \tag{6.1}$$

The terms in the integrand are respectively the matter kinetic term, gauge kinetic term and the Fayet-Iliopoulos-Theta term, which are given by

$$\begin{aligned} L_{kin} \;=\; & -D^\mu \overline{\phi} D_\mu \phi + \frac{i}{2} \overline{\psi}_- (\overleftrightarrow{D_0} + \overleftrightarrow{D_1}) \psi_- + \frac{i}{2} \overline{\psi}_+ (\overleftrightarrow{D_0} - \overleftrightarrow{D_1}) \psi_+ + D|\phi|^2 + |F|^2 \\ & - |\sigma|^2 |\phi|^2 - \overline{\psi}_- \sigma \psi_+ - \overline{\psi}_+ \overline{\sigma} \psi_- - i\overline{\phi} \lambda_- \psi_+ + i\overline{\phi} \lambda_+ \psi_- + i\overline{\psi}_+ \overline{\lambda}_- \phi - i\overline{\psi}_- \overline{\lambda}_+ \phi, \end{aligned} \tag{6.2}$$

$$L_{gauge} \;=\; \frac{1}{2e^2} \left[-\partial^\mu \overline{\sigma} \partial_\mu \sigma + \frac{i}{2} \overline{\lambda}_- (\overleftrightarrow{\partial}_0 + \overleftrightarrow{\partial}_1) \lambda_- + \frac{i}{2} \overline{\lambda}_+ (\overleftrightarrow{\partial}_0 - \overleftrightarrow{\partial}_1) \lambda_+ + v_{01}^2 + D^2 \right], \tag{6.3}$$

$$L_{FI,\theta} \;=\; -rD + \theta v_{01}. \tag{6.4}$$

In the above expressions, the notation $\overline{\psi} \overleftrightarrow{D_\mu} \psi = \overline{\psi}(D_\mu \psi) - (D_\mu \overline{\psi})\psi$ is used. Also, we have written the Lagrangian only in the case of single matter field of unit charge ($N = 1$, $Q_1 = 1$) to avoid complicated expressions, but the generalization is obvious.

If there were no boundary of Σ, the system would be invariant under $(2, 2)$ supersymmetry whose transformation laws are given by

$$\begin{aligned} &\delta v_\pm = i\overline{\epsilon}_\pm \lambda_\pm + i\epsilon_\pm \overline{\lambda}_\pm, \\ &\delta \sigma = -i\overline{\epsilon}_+ \lambda_- - i\epsilon_- \overline{\lambda}_+, \\ &\delta D = \frac{1}{2} \Big(-\overline{\epsilon}_+ (\partial_0 - \partial_1) \lambda_+ - \overline{\epsilon}_- (\partial_0 + \partial_1) \lambda_- + \epsilon_+ (\partial_0 - \partial_1) \overline{\lambda}_+ + \epsilon_- (\partial_0 + \partial_1) \overline{\lambda}_- \Big). \\ &\delta \lambda_+ = i\epsilon_+ (D + iv_{01}) + \epsilon_- (\partial_0 + \partial_1) \overline{\sigma}, \\ &\delta \lambda_- = i\epsilon_- (D - iv_{01}) + \epsilon_+ (\partial_0 - \partial_1) \sigma, \end{aligned} \tag{6.5}$$

and

$$\begin{aligned} &\delta \phi = \epsilon_+ \psi_- - \epsilon_- \psi_+, \\ &\delta \psi_+ = i\overline{\epsilon}_- (D_0 + D_1) \phi + \epsilon_+ F - \overline{\epsilon}_+ \overline{\sigma} \phi, \\ &\delta \psi_- = -i\overline{\epsilon}_+ (D_0 - D_1) \phi + \epsilon_- F + \overline{\epsilon}_- \sigma \phi, \\ &\delta F = -i\overline{\epsilon}_+ (D_0 - D_1) \psi_+ - i\overline{\epsilon}_- (D_0 + D_1) \psi_- + (\overline{\epsilon}_+ \overline{\sigma} \psi_- + \overline{\epsilon}_- \sigma \psi_+) + i(\overline{\epsilon}_- \overline{\lambda}_+ - \overline{\epsilon}_+ \overline{\lambda}_-) \phi. \end{aligned} \tag{6.6}$$

In the case where Σ has a boundary (where we now consider the strip $\Sigma = \mathbf{R} \times I$), the action transforms under (6.5) and (6.6) as

$$\begin{aligned}\delta S = \frac{1}{4\pi}\int_{\partial\Sigma} \mathrm{d}x^0 \Big\{ & \epsilon_+\Big[\, T\overline{\lambda}_+ - (D_0+D_1)\overline{\phi}\psi_- + i\overline{\phi}\sigma\psi_+ + \frac{i}{2e^2}\overline{\lambda}_-(\partial_0+\partial_1)\sigma + i\overline{\psi}_+ F\,\Big] \\ & + \epsilon_-\Big[-\overline{T}\overline{\lambda}_- - (D_0-D_1)\overline{\phi}\psi_+ + i\overline{\phi}\overline{\sigma}\psi_- - \frac{i}{2e^2}\overline{\lambda}_+(\partial_0-\partial_1)\overline{\sigma} - i\overline{\psi}_- F\Big] \\ & + \overline{\epsilon}_+\Big[-\overline{T}\lambda_+ + \overline{\psi}_-(D_0+D_1)\phi + i\overline{\psi}_+\overline{\sigma}\phi + \frac{i}{2e^2}(\partial_0+\partial_1)\overline{\sigma}\lambda_- + i\overline{F}\psi_+\Big] \\ & + \overline{\epsilon}_-\Big[\, T\lambda_- + \overline{\psi}_+(D_0-D_1)\phi + i\overline{\psi}_-\sigma\phi - \frac{i}{2e^2}(\partial_0-\partial_1)\sigma\lambda_+ - i\overline{F}\psi_-\Big] \Big\},\end{aligned} \tag{6.7}$$

where T is defined by

$$T = \left(r - |\phi|^2 - \frac{D}{2e^2}\right) - i\left(\theta + \frac{v_{01}}{2e^2}\right). \tag{6.8}$$

We look for a boundary condition such that B-type supersymmetry generated by $Q = \overline{Q}_+ + \mathrm{e}^{i\beta}\overline{Q}_-$ and $Q^\dagger$ is unbroken.

Before discussing the detail, we note that the locality of equation motion for the gauge fields requires the boundary condition

$$\frac{v_{01}}{e^2} = -\theta, \quad \text{at } \partial\Sigma. \tag{6.9}$$

Also, the auxiliary fields are solved by $F = 0$ and

$$\frac{D}{e^2} = r - |\phi|^2. \tag{6.10}$$

If we use these relations we have $T = \frac{1}{2}(r - |\phi|^2 - i\theta) = \frac{1}{2e^2}(D + iv_{01})$ at $\partial\Sigma$.

We also make an important remark. Since $v_{01} = \partial_0 v_1 - \partial_1 v_0$ is a total derivative, the Theta term of the action can naïvely be written as the boundary term

$$\frac{\theta}{2\pi}\int_{\Sigma} v_{01}\mathrm{d}^2x \stackrel{?}{=} -\frac{\theta}{2\pi}\int_{\partial\Sigma} v_0 \mathrm{d}x^0. \tag{6.11}$$

However, v_0 is not gauge invariant whereas v_{01} is. In particular, when the boundary components are compactified on circles, the right hand side changes by integer multiples of θ under gauge transformations. Thus, for a generic θ, (6.11) is not an allowed thing to do. If θ is an integer multiple of 2π, however, the right hand side of (6.11) is gauge invariant up to 2π shifts so that $\exp\left(i\frac{\theta}{2\pi}\oint v\right)$ is well-defined. Thus, only for those cases, the manipulation (6.11) is allowed. More generally, for a general θ one can shift $\theta \to \theta - 2\pi n$, with n integer, provided the boundary term $-n\int_{\partial\Sigma} v_0 \mathrm{d}x^0$ is added to the action.

Pure Maxwell Theory

We start the study with the simplest case; without matter. In this case, the theory has a single twisted chiral (gauge) multiplet Σ with the twisted superpotential

$$\widetilde{W} = -t\Sigma, \tag{6.12}$$

where t is the complex combination of the FI and Theta parameters

$$t = r - i\theta. \tag{6.13}$$

B-type boundary condition for twisted chiral multiplet fields is like A-type boundary condition for chiral multiplet fields. In particular, the world sheet boundary must end on a middle dimensional Lagrangian submanifold whose image in the $\widetilde{W}$-plane is a straight line. Since (6.12) is linear in Σ, this means that the worldsheet boundary must end on a straight line in Σ. The Lagrangian condition is trivially satisfied. Thus, if we denote the phase of the FI-Theta parameter t as

$$t = |t|\mathrm{e}^{i\gamma}, \tag{6.14}$$

the B-type supersymmetric D-brane is the straight line in the σ-plane whose slope is given by $-\gamma$;

$$\mathrm{Im}(\mathrm{e}^{i\gamma}\sigma) = \text{constant} \quad \text{at} \;\; \partial\Sigma. \tag{6.15}$$

The boundary condition on the component fields is given by

$$\begin{aligned}
&\mathrm{e}^{i\gamma}(\partial_0+\partial_1)\sigma = \mathrm{e}^{-i\gamma}(\partial_0-\partial_1)\overline{\sigma}, \\
&\mathrm{e}^{-i\gamma}\lambda_+ + \mathrm{e}^{i\gamma}\lambda_- = 0, \qquad\qquad \text{at} \;\; \partial\Sigma. \\
&\mathrm{e}^{i\gamma}\overline{\lambda}_+ + \mathrm{e}^{-i\gamma}\overline{\lambda}_- = 0,
\end{aligned} \tag{6.16}$$

It is indeed easy to check that the variation (6.7) vanishes for B-type supersymmetry with $\epsilon_- = -\epsilon_+$. For supersymmetry with $\epsilon_- = -\mathrm{e}^{i\beta}\epsilon_+$, we only have to make the replacement $\sigma \to \mathrm{e}^{-i\beta}\sigma$, $\lambda_\pm \to \mathrm{e}^{\pm i\beta/2}\lambda_\pm$ in the above expressions.

The zero point energies of σ and $\lambda_\pm$ cancel against each other and the vacuum energy of the system comes purely from the gauge field sector. By the equation of motion (or a Gauss law constraint) $\partial_1 v_{01} = 0$, the field strength v_{01} is a constant and by the boundary condition (6.9) it is given by $v_{01} = -e^2\theta$. The vacuum energy is then given by

$$E_0 = \pi\frac{e^2|t|^2}{2}. \tag{6.17}$$

In particular, the supersymmetry is spontaneously broken if $t \neq 0$ as can be seen also by the supersymmetry transformation of $\lambda_\pm$ in (6.5) (where $D \pm iv_{01} = e^2(r \mp i\theta)$ by the constraint). All these are the same as the elements of the standard story in the bulk theory.

The General Case

Let us now consider the case with matters. It is known that, under certain conditions, the bulk theory can be identified as a non-linear sigma model at low enough energies compared to $e\sqrt{r}$ (see for example [42, 43, 2]). The target space X is a toric manifold defined as the solution space to $\sum_{i=1}^{N} Q_i|\phi_i|^2 = r$ modded out by the $U(1)$ gauge transformations. We look for the boundary conditions corresponding to D-branes wrapping totally on X (with or without coupling to gauge fields on X).

Theta angle in the gauge theory is usually identified as the B-field. In non-linear sigma models, as we have seen in section 3, B-field modifies the boundary condition on the coordinate fields as (3.21), from pure Neumann to mixed Dirichlet-Neumann condition. However, in the gauge theory with action (6.1), the condition remains pure Neumann type $D_1\phi = 0$ even if we turn on θ. Thus, there appears to be a discrepancy between the gauge theory and the non-linear sigma model when formulated on a worldsheet with boundary. This mismatch can be cured by adding the boundary term

$$S_{boundary} = \frac{\theta}{4\pi r}\int_{\partial\Sigma}\Big(iD_0\overline{\phi}\,\phi - i\overline{\phi}D_0\phi\Big)\,\mathrm{d}x^0 \tag{6.18}$$

to the action (6.1). Then, the boundary condition required from the locality of equation of motion becomes

$$\cos(\gamma)\,D_1\phi - i\sin(\gamma)\,D_0\phi = 0, \quad \text{at} \;\; \partial\Sigma, \tag{6.19}$$

where $t = r - i\theta = |t|\mathrm{e}^{i\gamma}$. This corresponds to the mixed Dirichlet-Neumann boundary condition of (3.21). We note that the addition of (6.18) also alters the boundary condition (6.9) for the gauge field as

$$\frac{v_{01}}{e^2} = -\theta + \frac{|\phi|^2}{r}\theta. \tag{6.20}$$

The total action

$$S_{\rm tot} = S + S_{boundary} \tag{6.21}$$

transforms under (6.5) and (6.6) as

$$\begin{aligned}\delta S_{\rm tot} = \frac{1}{4\pi}\int_{\partial\Sigma}\mathrm{d}x^0\,\Big\{\; &\epsilon_+\Big[\,\widetilde{T}\overline{\lambda}_+ - \Big((1-\tfrac{2i\theta}{r})D_0 + D_1\Big)\overline{\phi}\psi_- + i\overline{\phi}\sigma\psi_+ + \frac{i}{2e^2}\overline{\lambda}_-(\partial_0+\partial_1)\sigma\,\Big]\\ &+\epsilon_-\Big[-\overline{\widetilde{T}}\overline{\lambda}_- - \Big((1+\tfrac{2i\theta}{r})D_0 - D_1\Big)\overline{\phi}\psi_+ + i\overline{\phi}\overline{\sigma}\psi_- - \frac{i}{2e^2}\overline{\lambda}_+(\partial_0-\partial_1)\overline{\sigma}\Big]\\ &+\overline{\epsilon}_+\Big[-\overline{\widetilde{T}}\lambda_+ + \overline{\psi}_-\Big((1+\tfrac{2i\theta}{r})D_0 + D_1\Big)\phi + i\overline{\psi}_+\overline{\sigma}\phi + \frac{i}{2e^2}(\partial_0+\partial_1)\overline{\sigma}\lambda_-\Big]\\ &+\overline{\epsilon}_-\Big[\,\widetilde{T}\lambda_- + \overline{\psi}_+\Big((1-\tfrac{2i\theta}{r})D_0 - D_1\Big)\phi + i\overline{\psi}_-\sigma\phi - \frac{i}{2e^2}(\partial_0-\partial_1)\sigma\lambda_+\Big]\;\Big\}.\end{aligned} \tag{6.22}$$

Here $\widetilde{T}$ is given by

$$\begin{aligned}\widetilde{T} &= \left(r - |\phi|^2 - \frac{D}{2e^2}\right) - i\left(\theta\Big(1 - \frac{|\phi|^2}{r}\Big) + \frac{v_{01}}{2e^2}\right)\\ &= \frac{r - |\phi|^2}{2r}\Big(r - i\theta\Big),\end{aligned} \tag{6.23}$$

where we have used (6.10) and the new boundary condition (6.20) in the second equality. Since $\widetilde{T}$ is proportional to $t = r - i\theta$ as in the pure Maxwell theory, it is obvious that Φ independent part of the variation (6.22) vanishes for $\epsilon_- = -\epsilon_+$ under the same condition (6.16) as in the Maxwell theory. We are now left with the following terms (for $\epsilon_- = -\epsilon_+$)

$$\begin{aligned}&\epsilon_+\Big[-\Big((1 - \tfrac{2i\theta}{r})D_0 + D_1\Big)\overline{\phi}\psi_- + \Big((1 + \tfrac{2i\theta}{r})D_0 - D_1\Big)\overline{\phi}\psi_+ + i\overline{\phi}(\sigma\psi_+ - \overline{\sigma}\psi_-)\Big]\\ &+\overline{\epsilon}_+\Big[\,\overline{\psi}_-\Big((1 + \tfrac{2i\theta}{r})D_0 + D_1\Big)\phi - \overline{\psi}_+\Big((1 - \tfrac{2i\theta}{r})D_0 - D_1\Big)\phi + i(\overline{\psi}_+\overline{\sigma} - \overline{\psi}_-\sigma)\phi\Big].\end{aligned}$$

The non-derivative terms vanish if the straight line of σ is of the type:

$$\mathrm{Im}(\mathrm{e}^{i\gamma}\sigma) = 0 \quad \text{at} \quad \partial\Sigma, \tag{6.24}$$

and the matter fermions satisfy the boundary condition

$$\begin{aligned}\mathrm{e}^{-i\gamma}\psi_+ &= \mathrm{e}^{i\gamma}\psi_-,\\ \mathrm{e}^{i\gamma}\overline{\psi}_+ &= \mathrm{e}^{-i\gamma}\overline{\psi}_-,\end{aligned} \qquad \text{at} \quad \partial\Sigma. \tag{6.25}$$

It is now straightforward to see that the derivative terms also vanish under the boundary conditions (6.19) and (6.25). It is also easy to see that these boundary conditions (including (6.24)) are invariant under the B-type supersymmetry.

To summarize, the total action $S_{\rm tot}$ has B-type supersymmetry with $\epsilon_- = -\epsilon_+$ under the boundary conditions (6.19) and (6.25) for the matter fields and (6.16), (6.24), and (6.20) for the gauge multiplet fields. These conditions reduce to the ordinary mixed Dirichlet-Neumann boundary conditions (3.21) and (3.22) of the non-linear sigma model on X. To recover the phase, $\epsilon_- = -\mathrm{e}^{i\beta}\epsilon_+$, it is enough to make the replacement $\psi_\pm \to \mathrm{e}^{\pm i\beta/2}\psi_\pm$, $\sigma \to \mathrm{e}^{-i\beta}\sigma$ and $\lambda_\pm \to \mathrm{e}^{\pm i\beta/2}\lambda_\pm$.

So far we have been analyzing the boundary condition of the classical theory. There are two important quantum effects of the theory with $\sum_{i=1}^N Q_i \neq 0$; the running of the FI parameter r and the anomaly of the axial $U(1)$ R-symmetry. From the running of r, $r_0 = \sum_{i=1}^N Q_i \log(\Lambda_{\rm UV}/\Lambda)$, the phase $\mathrm{e}^{i\gamma} = t/|t|$ which enters in the boundary condition changes along the renormalization group flow. In particular, if $\sum_{i=1}^N Q_i > 0$ (which corresponds to an asymptotic free sigma model), the "bare phase" becomes trivial $\mathrm{e}^{i\gamma_0} \to 1$ in the continuum limit $\Lambda_{\rm UV}/\Lambda \to \infty$. Also, by the axial anomaly, the axial rotation can be done

not just by the replacement $\psi_\pm \to \mathrm{e}^{\pm i\beta/2}\psi_\pm$, $\sigma \to \mathrm{e}^{-i\beta}\sigma$ and $\lambda_\pm \to \mathrm{e}^{\pm i\beta/2}\lambda_\pm$ but *together with the shift of the Theta angle* $\theta \to \theta + \sum_{i=1}^N Q_i\beta$. These effects should be visible in a quantum effective description. Here we look at the effective action in terms of Σ-field whose scalar component is chosen to have large expectation values. This is obtained by integrating out the charged matter fields and is given (for $Q_i = 1$ case) by

$$\widetilde{W} = -N\Sigma(\log\Sigma - 1) - t\Sigma. \tag{6.26}$$

This yields the following effective FI-Theta parameter

$$t_{\mathit{eff}} = t + N\log\Sigma, \tag{6.27}$$

where the energy scale is set by the value of Σ. This effective theory is essentially the LG model with the superpotential (6.26)[1] which has N non-degenerate critical points $\Sigma_a = \mathrm{e}^{-t/N+2\pi ai/N}$ $(a = 0, \ldots, N-1)$. As we have seen, a D-brane preserving the B-type supercharge $Q = \overline{Q}_+ + \overline{Q}_-$ is the preimage of the straight line in the $\widetilde{W}$-plane. The equation is given by

$$\mathrm{Im}\Big(\mathrm{e}^{i\gamma_{\mathit{eff}}(\sigma)}\sigma\Big) = \text{constant}, \tag{6.28}$$

where $t_{\mathit{eff}} - N = \mathrm{e}^{i\gamma_{\mathit{eff}}}|t_{\mathit{eff}} - N|$. If we insist the straight line to pass through a critical value $\widetilde{W}(\Sigma_a) \sim N\mathrm{e}^{-t/N}$, the constant in the r.h.s. is of order $\mathrm{e}^{-t/N}$ and can be considered as the correction to the condition (6.24). It is in general a non-trivial task to find the explicit solution to the straight line equation. However, there is a trivial one if the Theta angle vanishes $\theta = 0$. In this case $\sigma = |\sigma|$ is a solution to the straight line equation with the zero slope $\mathrm{e}^{i\beta} = 1$. By the axial rotation $\sigma \to \mathrm{e}^{-i\beta}\sigma$, $\lambda_\pm \to \mathrm{e}^{\pm i\beta/2}\lambda_\pm$, we obtain the solution $\sigma = \mathrm{e}^{i\beta}|\sigma|$ with the slope β. However, we should note that this axial rotation shifts the Theta angle from zero to $\theta = N\beta$. Indeed the image of $\sigma = \mathrm{e}^{i\beta}|\sigma|$ in the $\widetilde{W}$-plane is a straight line only when this shift is made. Thus, we have seen that there is a one parameter family of explicit solutions

$$\begin{aligned}
&\sigma = \mathrm{e}^{i\theta/N}|\sigma|, \\
&\mathrm{e}^{i\theta/2N}\lambda_+ + \mathrm{e}^{-i\theta/2N}\lambda_- = 0, \qquad \text{at} \;\; \partial\Sigma, \\
&\mathrm{e}^{-i\theta/2N}\overline{\lambda}_+ + \mathrm{e}^{i\theta/2N}\overline{\lambda}_- = 0,
\end{aligned} \tag{6.29}$$

[1] Strictly speaking, the theory involves a gauge field. However, in the absence of light or tachyonic charged matter field, the effect of the gauge field is simply to create the vacuum energy $e^2(\mathrm{Im}\, t_{\mathit{eff}})^2/2$, as the standard auxiliary field does. There is actually a (minor) subtlety; If the theory is formulated on $\mathbf{R}^2$, the physics is periodic in θ which is identified as the constant electric field (divided by e^2). This is because of the pair creation of the electron and positron [44] which run away to opposite infinity in the space. However, if the theory is formulated on a strip, $\mathbf{R} \times [0, \pi]$, the electron positron pair, even if they are pair-created, can never run away to infinity. Thus, the physics is not periodic in θ.

parametrized by the worldsheet Theta angle θ. This preserves the supercharge

$$Q = \overline{Q}_+ + \mathrm{e}^{i\theta/N}\overline{Q}_- \tag{6.30}$$

and $Q^\dagger$. There are of course other solutions (especially those with $\beta \neq \theta/N$) but the quantum correction is non-trivial and it is not easy to determine them explicitly. It is easy to extend the above solutions to the general Q_i's: replace N in these formulae by $\sum_{i=1}^N Q_i$.

There is actually a better quantum effective description of the bulk theory found in [2], using the dual variables Y_i of the charged fields Φ_i. Later in this section and further in the next section, we will see how the boundary condition is described in that theory. This will lead to the map of D-branes under mirror symmetry.

Coupling to Gauge Fields on X

So far, we have been considering a gauge theory that corresponds to the non-linear sigma model on X with a B-field, but not including coupling the worldsheet boundary to the target space gauge fields. Now it is useful to observe that the B-field obeying a certain quantization condition can be considered as the curvature of a gauge field A_I on X. In such a case, as noted in section 3, the coupling to B field is equal to the boundary coupling to the gauge field A_I. The quantized B field corresponds to the case where the worldsheet Theta angle becomes an integer multiple of 2π, $\theta = 2\pi n$. We now recall that in such a case (and only in such a case) the worldsheet Theta term can be converted into a boundary term (6.11). Then, the total boundary term becomes

$$\begin{aligned} S'_{\text{boundary}} &= \frac{n}{2r}\int_{\partial\Sigma}\Big(iD_0\overline{\phi}\,\phi - i\overline{\phi}D_0\phi\Big)\,\mathrm{d}x^0 - n\int_{\partial\Sigma} v_0\,\mathrm{d}x^0 \\ &= \frac{n}{2r}\int_{\partial\Sigma}\Big(i\partial_0\overline{\phi}\,\phi - i\overline{\phi}\partial_0\phi + 2v_0(|\phi|^2 - r)\Big)\,\mathrm{d}x^0 \end{aligned} \tag{6.31}$$

In the sigma model limit $e\sqrt{r} \to \infty$, the constraint $|\phi|^2 = r$ is strictly imposed. Then, the boundary term is given by

$$S'_{\text{boundary}} = -n\int_{\partial\Sigma} A_I\partial_0\phi^I\mathrm{d}x^0 \tag{6.32}$$

where

$$A_I\mathrm{d}\phi^I = \frac{i}{2}\frac{\sum_{i=1}^N\overline{\phi}_i\overleftrightarrow{\mathrm{d}}\phi_i}{\sum_{i=1}^N Q_i|\phi_i|^2}. \tag{6.33}$$

In this expression, we have recovered all the N matter fields of charge $Q_1, \ldots, Q_N$ where the constraint is $\sum_{i=1}^N Q_i|\phi_i|^2 = r$.

The gauge field A_I in (6.33) is nothing but the hermitian connection of the natural holomorphic line bundle $\mathcal{O}_X(1)$ on the toric manifold X (where ϕ_i's represent the sections) with respect to the natural hermitian metric. To see this, let us make a gauge transformation $\phi_i \to \mathrm{e}^{iQ_i\lambda}\phi_i$. Then, the gauge field transforms as

$$A_I \mathrm{d}\phi^I \to A_I \mathrm{d}\phi^I - \mathrm{d}\lambda. \tag{6.34}$$

This is indeed the transformation property of a connection form of the bundle $\mathcal{O}_X(1)$. For example, let us consider the simplest case $X = \mathbf{C}\mathrm{P}^1$ where the gauge theory has two matters Φ_1, Φ_2 of charge 1. In the gauge where $\phi_1 = 1$ and $\phi_2 = z$, the gauge field (6.33) is given by

$$A = \frac{i}{2}\frac{\overline{z}\mathrm{d}z - z\mathrm{d}\overline{z}}{1+|z|^2}. \tag{6.35}$$

This is the gauge field of the line bundle $\mathcal{O}(1)$ of $\mathbf{C}\mathrm{P}^1$. Indeed, the first Chern class is represented by the curvature $\frac{i}{2\pi} \cdot i\mathrm{d}A = \frac{i}{2\pi}\mathrm{d}z\mathrm{d}\overline{z}/(1+|z|^2)^2$ which is the positive unit volume form of $\mathbf{C}\mathrm{P}^1$.

Thus, we indeed see that the boundary term (6.31) corresponds to the boundary coupling to the natural gauge fields of the bundle $\mathcal{O}_X(-n)$. Here we have to bear in mind that the boundary condition should be given by (6.19)-(6.25) and (6.16)-(6.24)-(6.20) where it is understood that $\gamma = \arg(r - 2\pi n i)$. If we turn on the bulk θ-term anew, the angle is given by $\gamma = \arg(t - 2\pi n i)$ where $t = r - i\theta$.

Alternative Formulation

In the non-linear sigma model, we have seen that there is an alternative formulation for coupling to target space gauge fields where we do not change the boundary condition but add a fermion bilinear boundary term. This can also be done in the gauge theory. The relevant boundary term for the gauge field of the bundle $\mathcal{O}_X(-n)$ is given by

$$S''_{\mathrm{boundary}} = \frac{n}{2r}\int\limits_{\partial\Sigma}\Big(iD_0\overline{\phi}\,\phi - i\overline{\phi}D_0\phi + (\psi_+ + \psi_-)(\overline{\psi}_+ + \overline{\psi}_-) - (\sigma+\overline{\sigma})|\phi|^2\Big)\,\mathrm{d}x^0. \tag{6.36}$$

It is straightforward to check that this is by itself invariant under the B-type supersymmetry with $\epsilon_- = -\epsilon_+$. Thus, one can add this to the total action S_{tot} without changing the boundary condition. We note that, as in non-linear sigma models, the equations of motion for the worldsheet fields have boundary contributions in this formulation.

6.2 A Review of a Derivation of Mirror Symmetry

We now briefly review the dual description of the gauge theory found in [2].

Let us consider the $U(1)$ gauge theory on $\Sigma = \mathbf{R}^2$ with matters of charge $Q_1, \ldots, Q_N$. The action is given by (6.1) (in the case $N=1, Q_1=1$). In the superfield notation, the Lagrangian is expressed as

$$L = \int \mathrm{d}^4\theta \left(\sum_{i=1}^{N} \overline{\Phi}_i \,\mathrm{e}^{2Q_iV} \Phi_i - \frac{1}{2e^2}\overline{\Sigma}\Sigma \right) + \frac{1}{2}\Big(-\int \mathrm{d}^2\widetilde{\theta}\; t\,\Sigma \,+\, c.c. \Big). \tag{6.37}$$

If we dualize the phase of the charged chiral superfield Φ_i, we obtain a neutral twisted chiral superfield Y_i that is periodic with periodicity $2\pi i$, $Y_i \equiv Y_i + 2\pi i$. The fields Y_i are related to the original charged chiral superfields Φ_i by

$$Y_i + \overline{Y}_i = 2\overline{\Phi_i}\mathrm{e}^{2Q_iV}\Phi_i, \tag{6.38}$$

or in components, $Y_i = y_i + \sqrt{2}\theta^+\overline{\chi}_{i+} + \sqrt{2}\overline{\theta}^-\chi_{i-} + \cdots$,

$$\begin{aligned}
&y_i = \varrho_i - i\vartheta_i, \quad \begin{cases} \varrho_i = |\phi_i|^2, \\ \partial_\pm\vartheta_i = \pm 2\Big(-|\phi_i|^2(\partial_\pm\varphi_i + Q_i v_\pm) + \overline{\psi}_{i\pm}\psi_{i\pm}\Big), \end{cases} \\
&\chi_{i+} = 2\overline{\psi}_{i+}\phi_i, \quad \chi_{i-} = -2\overline{\psi}_{i-}\phi_i, \\
&\overline{\chi}_{i+} = 2\overline{\phi}_i\psi_{i+}, \quad \overline{\chi}_{i-} = -2\overline{\phi}_i\psi_{i-},
\end{aligned} \tag{6.39}$$

where φ_i is the phase of ϕ_i, $\phi_i = |\phi_i|\,\mathrm{e}^{i\varphi_i}$.

The fields Y_i couple to the gauge field as dynamical Theta angle. Thus, at the level of dualization we have the twisted superpotential $\widetilde{W} = \Sigma(\sum_{i=1}^N Q_iY_{i0} - t_0)$ where subscript 0 stands for the bare parameters and fields. The FI parameter runs as $r_0 = b_1\log(\Lambda_{UV}/\Lambda)$ with

$$b_1 := \sum_{i=1}^{N} Q_i, \tag{6.40}$$

but one can make the superpotential finite by renormalizing the bare fields ϱ_{i0} as $\varrho_{i0} = \varrho_i + \log(\Lambda_{UV}/\mu)$, where μ is the renormalization point. The Kähler metric of the y_i variables is given classically by

$$\mathrm{d}s^2 = \sum_{i=1}^{N} \frac{|\mathrm{d}y_i|^2}{2(2r_0/b_1 + y_i + \overline{y}_i)} \simeq \frac{b_1}{4r_0}\sum_{i=1}^{N}|\mathrm{d}y_i|^2. \tag{6.41}$$

This superpotential is corrected by instanton effect where the instantons are the vortices of the gauge theory. The correction is of the form e^{-Y_i} and the exact twisted superpotential is given by

$$\widetilde{W} = \Sigma\left(\sum_{i=1}^{N} Q_iY_i - t(\mu) \right) + \sum_{i=1}^{N} \mu\,\mathrm{e}^{-Y_i}. \tag{6.42}$$

In the case $b_1 \neq 0$, the FI parameter is renormalized and $\Lambda = \mu e^{-t/b_1}$ is renormalization group invariant, as can be seen also from (6.42) using the shifts of Y_i's. In the conformal case $b_1 = 0$, t is the dimensionless parameter of the theory, and μ can be simply absorbed by the shifts of Y_i's. In what follows, we omit the scale μ.

In the sigma model limit $e\sqrt{r} \to \infty$, the gauge multiplet fields becomes infinitely heavy and can be integrated out. Then, this yields a constraint

$$\sum_{i=1}^{N} Q_i Y_i = t. \tag{6.43}$$

Thus, we obtain a theory of N periodic fields Y_i with one constraint (6.43) which has a twisted superpotential

$$\widetilde{W} = \sum_{i=1}^{N} e^{-Y_i}. \tag{6.44}$$

In other words, we obtain a LG model on $(\mathbf{C}^\times)^{N-1}$. Since the original gauge theory becomes non-linear sigma model on the toric manifold X in the limit $e\sqrt{r} \to \infty$, the above LG model is a dual description of the non-linear sigma model on X. Since it is described by twisted chiral fields, it is the mirror of the sigma model on X.

It is easy to find the critical point of the superpotential (6.44) under the constraint (6.43). There are b_1 critical points $p_0, \ldots, p_{b_1-1}$, where at the a-th critical point $e^{-y_i}(p_a) = Q_i e^{-t/b_1 + 2\pi a i/b_1} \prod_{j=1}^{N} Q_j^{-Q_j}$ with the critical value

$$\widetilde{w}_a = b_1 e^{-t/b_1 + 2\pi a i/b_1} \prod_{j=1}^{N} Q_j^{-Q_j}. \tag{6.45}$$

All these are massive vacua at which the $\mathbf{Z}_{2b_1}$ axial R-symmetry is spontaneously broken to $\mathbf{Z}_2$.

6.3 D-branes and Mirror Symmetry: First Example

We would like to see how the boundary conditions for the linear sigma model can be described in the quantum effective theory in terms of the dual variables. We consider the model with

$$b_1 = \sum_{i=1}^{N} Q_i > 0, \tag{6.46}$$

that corresponds to an asymptotic free non-linear sigma model. Since the dual theory is a LG model described in terms of twisted chiral superfields, B-type supersymmetry looks like A-type supersymmetry for chiral superfields. In particular, the worldsheet boundary

must end on a middle dimensional Lagrangian submanifold of $(\mathbf{C}^{\times})^{N-1}$ that is mapped to a straight line in the $\widetilde{W}$-plane.

We focus on the family of boundary conditions (6.29) parametrized by the worldsheet Theta angle θ. The boundary conditions on the matter fields are

$$\begin{aligned}
&\cos(\gamma_0)\, D_1\phi_i - i\sin(\gamma_0)\, D_0\phi_i = 0, \\
&\mathrm{e}^{-i\gamma_0+i\theta/2b_1}\psi_{i+} = \mathrm{e}^{i\gamma_0-i\theta/2b_1}\psi_{i-}, \qquad \text{at } \partial\Sigma. \\
&\mathrm{e}^{i\gamma_0-i\theta/2b_1}\overline{\psi}_{i+} = \mathrm{e}^{-i\gamma_0+i\theta/2b_1}\overline{\psi}_{i-},
\end{aligned} \tag{6.47}$$

We note that the phase $\mathrm{e}^{i\gamma_0}$ defined by $r_0 - i\theta = \mathrm{e}^{i\gamma_0}|r_0 - i\theta|$ becomes trivial

$$\gamma_0 \to 0, \tag{6.48}$$

in the continuum limit $\Lambda_{\rm UV} \to \infty$ where $r_0 = b_1\log(\Lambda_{\rm UV}/\Lambda) \to \infty$. This boundary condition preserves the supercharge

$$Q = \overline{Q}_+ + \mathrm{e}^{i\theta/b_1}\overline{Q}_- \tag{6.49}$$

and its conjugate $Q^\dagger$.

We recall that there is a boundary term in the action

$$\begin{aligned}
S_{boundary} &= \frac{\theta}{4\pi r_0}\int_{\partial\Sigma}\sum_{i=1}^{N}\Big(iD_0\overline{\phi}_i\,\phi_i - i\overline{\phi}_i D_0\phi_i\Big)\,\mathrm{d}x^0 \\
&= \frac{\theta}{2\pi r_0}\int_{\partial\Sigma}\sum_{i=1}^{N}|\phi_i|^2(\partial_0\varphi_i + Q_i v_0)\,\mathrm{d}x^0.
\end{aligned} \tag{6.50}$$

Note that in terms of the renormalized dual fields we have $|\phi_i|^2 = r_0/b_1 + \varrho_i$. Then, in the continuum limit the boundary term can be written as

$$S_{boundary} = \frac{\theta}{2\pi}\int_{\partial\Sigma}\left(\frac{1}{b_1}\sum_{i=1}^{N}\partial_0\varphi_i \,+\, v_0\right)\mathrm{d}x^0. \tag{6.51}$$

Now the relevant part of the action in the dualization is

$$S_\varphi = \frac{1}{2\pi}\int_\Sigma\sum_{i=1}^{N}r_0^2|\mathrm{d}\varphi_i + Q_i v|^2 - \frac{i\theta}{2\pi}\int_{\partial\Sigma}\Big(\frac{1}{b_1}\sum_{i=1}^{N}\mathrm{d}\varphi_i \,+\, v\Big) \tag{6.52}$$

where we consider Euclidean signature (for simplicity) and we ignore the fermionic components which are not essential in this part of the argument. We consider another system involving one-form fields $\mathcal{B}_i = \mathcal{B}_{i\mu}\mathrm{d}x^\mu$ with the action given by

$$S' = \sum_{i=1}^{N}\left[\frac{1}{8\pi r_0^2}\int_\Sigma \mathcal{B}_i\wedge *\mathcal{B}_i + \frac{i}{2\pi}\int_\Sigma \mathcal{B}_i\wedge(\mathrm{d}\varphi_i + Q_i v)\right] - \frac{i\theta}{2\pi}\int_{\partial\Sigma}\Big(\frac{1}{b_1}\sum_{i=1}^{N}\mathrm{d}\varphi_i \,+\, v\Big). \tag{6.53}$$

We require the boundary condition on the one-form fields $\mathcal{B}_i$ that they vanish against the tangent vectors of the boundary

$$\mathcal{B}_i|_{\partial\Sigma} = 0. \tag{6.54}$$

If we first integrate out the one-form field $\mathcal{B}_i$, we obtain the constraint $\mathcal{B}_i = i2r_0^2 * (\mathrm{d}\varphi_i + Q_i v)$ (which is consistent with the boundary condition (6.47) in the continuum limit) and we obtain the original action (6.52). Instead, if we first integrate out the variables φ_i, we obtain the constraint

$$\mathcal{B}_i = \mathrm{d}\vartheta_i \tag{6.55}$$

where ϑ_i are periodic variables of period 2π. By the boundary condition (6.54), we see that ϑ_i are constants along the boundary of $\partial\Sigma$. By the terms $-i(\theta/2\pi b_1)\int_{\partial\Sigma}\mathrm{d}\varphi_i$ in the action (6.53), we see that the constants are

$$\vartheta_i = \theta/b_1 \quad \text{at} \quad \partial\Sigma, \tag{6.56}$$

for all i. Now, if we plug the constraint (6.55) back into (6.53) we obtain the action

$$\begin{aligned} S_\vartheta &= \sum_{i=1}^{N}\left[\frac{1}{8\pi r_0^2}\int_\Sigma |\mathrm{d}\vartheta_i|^2 + \frac{i}{2\pi}\int_\Sigma \mathrm{d}\vartheta_i \wedge Q_i v\right] - \frac{i\theta}{2\pi}\int_{\partial\Sigma} v \\ &= \sum_{i=1}^{N}\left[\frac{1}{8\pi r_0^2}\int_\Sigma |\mathrm{d}\vartheta_i|^2 - \frac{i}{2\pi}\int_\Sigma Q_i\vartheta_i \mathrm{d}v\right] + \frac{i}{2\pi}\int_{\partial\Sigma}\Big(\sum_{i=1}^{N} Q_i\vartheta_i - \theta\Big)v, \end{aligned} \tag{6.57}$$

where we have performed partial integrations in the last step. Note that the boundary term in the right hand side vanishes if we use the boundary condition (6.56). Then, the dualization proceeds precisely as in the bulk theory and we will obtain the superpotential (6.42). In the sigma model limit $e^2\sqrt{r} \to \infty$, after integrating out the Σ-field, we obtain the constraint (6.43) and the twisted superpotential (6.44).

The boundary condition (6.56) means that e^{-y_i}'s at the boundary have a fixed common phase θ/b_1. Namely, the worldsheet boundary $\partial\Sigma$ is mapped by (e^{-y_i}) to a real $(N-1)$-dimensional cycle γ_θ in the algebraic torus $(\mathbf{C}^\times)^{N-1}$ defined by

$$(\mathrm{e}^{-y_1}, \ldots, \mathrm{e}^{-y_N}) = (\mathrm{e}^{-\varrho_1 + i\theta/b_1}, \ldots, \mathrm{e}^{-\varrho_N + i\theta/b_1}), \tag{6.58}$$

where $(\varrho_1, \ldots, \varrho_N)$ are the real coordinates constrained by $\sum_{i=1}^N Q_i\varrho_i = 0$. By the boundary condition for ϕ_i in (6.47), we see that the tangent coordinates ϱ_i obey the Neumann boundary condition

$$\partial_1\varrho_i = 0 \quad \text{at} \quad \partial\Sigma, \tag{6.59}$$

in the continuum limit. The boundary condition on the fermionic components can be read from (6.39) and (6.47) and is given by

$$\begin{aligned} e^{-i\theta/2b_1}\chi_{i+} + e^{i\theta/2b_1}\chi_{i-} &= 0, \\ e^{i\theta/2b_1}\overline{\chi}_{i+} + e^{-i\theta/2b_1}\overline{\chi}_{i-} &= 0, \end{aligned} \quad \text{at} \ \ \partial\Sigma. \tag{6.60}$$

These are the standard boundary condition on the worldsheet fields corresponding to the D-brane wrapped on γ_θ. The phases $e^{i\theta/2b_1}$ in the condition for the fermionic components shows that we are performing an R-rotation.

The cycle γ_θ is a Lagrangian submanifold of $(\mathbf{C}^\times)^{N-1}$ with respect to the flat cylinder metric (6.41). The image of the cycle γ_θ in the $\widetilde{W}$ plane is

$$\widetilde{W} = e^{i\theta/b_1}\sum_{i=1}^{N}|e^{-y_i}|, \tag{6.61}$$

and is indeed a straight line (see Figure 18). Moreover, the cycle passes through the

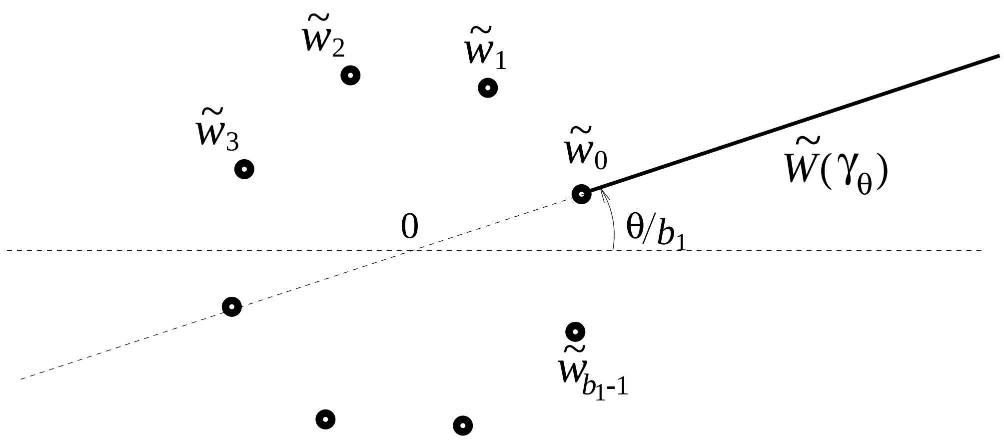

Figure 18: The image in the $\widetilde{W}$-plane of the cycle γ_θ.

critical point p_0 and the straight line in the $\widetilde{W}$-plane emanates outward from the critical value $\widetilde{w}_0$. Thus, γ_θ is the wavefront trajectory emanating from the critical point p_0. Since the image in the $\widetilde{W}$-plane has the slope θ/b_1 and the boundary condition of the fermions is rotated as (6.60), the boundary condition indeed preserves the supersymmetry Q and $Q^\dagger$ in (6.49).

When θ is an integer multiple of 2π, as we have noted before, in the non-linear sigma model limit the corresponding B-field is integral and the coupling of the worldsheet to such a B-field can be identified as the boundary coupling to the gauge field on X. In

particular, for $\theta = 2\pi n$, it is the gauge field of the bundle $\mathcal{O}_X(-n)$. Thus, the D-brane wrapped on $\gamma_{2\pi n}$ in the LG model can be considered as the mirror of the D-brane on X which supports the bundle $\mathcal{O}_X(-n)$ in the sigma model with trivial B-field. Note that the image of $\gamma_{2\pi n}$ in the $\widetilde{W}$-plane is the straight line emanating from $\widetilde{w}_n$ in the radial direction (where the labeling of the critical values is made in the theory with $\theta = 0$). More generally, the D-brane which supports the bundle $\mathcal{O}_X(-n)$ in the sigma model with the B-field corresponding to $\theta \neq 0$ is mirror to the D-brane wrapped on $\gamma_{\theta+2\pi n}$ whose image in the $\widetilde{W}$-plane is the straight line emanating from $\widetilde{w}_n$ in the radial direction.

In the next section, we make use of the connection explained in this section to determine the relation of the D-branes in the non-linear sigma model on the toric manifold X (including more general cases corresponding to the gauge group $U(1)^k$ in the linear sigma model) and the D-branes of the mirror Landau-Ginzburg model.

7 D-Branes and Mirror Symmetry: Massive Theories

In this section we will study how D-branes transform under mirror symmetry for sigma models on Kähler manifolds with $c_1 > 0$. We will mainly concentrate on the case where the theory has only massive vacua, and discuss the mirror of D-branes wrapped on holomorphic cycles on the target Kähler manifold X in terms of Lagrangian submanifolds of the mirror LG models. In particular we concentrate on D-branes which corresponds to exceptional bundles on X (to be defined below). It turns out that this connection explains the observations of Kontsevich noting a formal correspondence between the properties of Helices and exceptional bundles on the Kähler manifolds and the soliton numbers of an associated LG model.

In order to do this, we will first review what exceptional bundles and Helices are. Afterwards we discuss how mirror symmetry acts in this context.

7.1 D-branes, Exceptional bundles and Helices

The similarities between the structures appearing in the classification of $\mathcal{N} = 2$ theories [3] and Helix theory [45] was observed by Kontsevich [46]. This observation was the starting point of [16] in which the mysterious correspondence between the soliton numbers of a non-linear sigma model with $\mathbb{P}^N$ target space and exceptional bundles was further explored. We will explain this correspondence in this section as a consequence of mirror symmetry.

7.2 Exceptional bundles and mutations

A vector bundle or a sheaf E on an N-dimensional variety X with $c_1 > 0$ is called exceptional if [16, 45]

$$\mathrm{Ext}^0(E,E) = \mathbb{C}, \quad \mathrm{Ext}^i(E,E) = 0\,, \quad i \geq 1\,, \tag{7.1}$$

where Ext^i is the sheaf theory generalization of cohomology groups H^i, i.e., for vector bundles E and F, $\mathrm{Ext}^i(E,F) = H^i(X, E^* \otimes F)$ which in tern equals to the Dolbeault cohomology $H^{0,i}(X, E^* \otimes F)$. An exceptional collection is a collection of exceptional sheaves $\{E_1, \cdots, E_n\}$ such that if $a < b$ then [16, 45]

$$\begin{aligned} \mathrm{Ext}^i(E_a, E_b) &= 0\,,\ i \neq i_0 \qquad \text{for some } i_0\,, \\ \mathrm{Ext}^i(E_b, E_a) &= 0\,,\ i \geq 0\,. \end{aligned} \tag{7.2}$$

Note that the above condition leaves $\dim \mathrm{Ext}^{i_0}(E_a, E_b)$ undetermined for $a < b$ and that could in principle be any integer. The alternating sum of dimensions of the groups Ext^i defines a bilinear product [16, 45],

$$\chi(E,F) = \sum_{i=0}^{N} (-1)^i \dim_{\mathbb{C}} \mathrm{Ext}^i(E,F) = \int_X \mathrm{ch}(E^* \otimes F) \mathrm{Td}(X)\,. \tag{7.3}$$

An exceptional sheaf E has the property that $\chi(E,E) = 1$. An important property of an exceptional collection is that they can be transformed into new exceptional collections by transformations called mutations. For an exceptional collection of sheaves $\{E_1, \cdots, E_n\}$ we can sometimes define two transformations, left mutation and right mutation. Given a neighboring pair of sheaves (E_a, E_{a+1}) in an exceptional collection, the transformations L_{E_a} and R_{E_a} are such that

$$L_{E_a}(E_a, E_{a+1}) = (L_{E_a}(E_{a+1}), E_a)\,, \quad R_{a+1}(E_a, E_{a+1}) = (E_{a+1}, R_{E_{a+1}}(E_a))\,, \tag{7.4}$$

The transformed sheaf $L_{E_a}(E_{a+1})$ is defined through an exact sequence. The exact sequence used to define the mutated sheaf depends on the Ext^i groups of the pair (E_a, E_{a+1}) [45, 16],

• If $\mathrm{Ext}^0(E_a, E_{a+1}) \neq 0$ and $\mathrm{Ext}^0(E_a, E_{a+1}) \otimes E_a \mapsto E_{a+1}$ is surjective then $L_{E_a}(E_{a+1})$ is defined by the exact sequence,

$$0 \mapsto L_{E_a}(E_{a+1}) \mapsto \mathrm{Ext}^0(E_a, E_{a+1}) \otimes E_a \mapsto E_{a+1} \mapsto 0\,, \tag{7.5}$$

• If $\mathrm{Ext}^0(E_a, E_{a+1}) \neq 0$ and $\mathrm{Ext}^0(E_a, E_{a+1}) \otimes E_a \mapsto E_{a+1}$ is injective then $L_{E_a}(E_{a+1})$ is defined by the exact sequence,

$$0 \mapsto \mathrm{Ext}^0(E_a, E_{a+1}) \otimes E_a \mapsto E_{a+1} \mapsto L_{E_a}(E_{a+1}) \mapsto 0\,, \tag{7.6}$$

• If $\mathrm{Ext}^1(E_a, E_{a+1}) \neq 0$ then $L_{E_a}(E_{a+1})$ is defined by exact sequence

$$0 \mapsto E_{a+1} \mapsto L_{E_a}(E_{a+1}) \mapsto \mathrm{Ext}^1(E_a, E_{a+1}) \otimes E_a \mapsto 0\,. \tag{7.7}$$

Similarly we can define the right mutated sheaf $R_{E_{a+1}}(E_a)$ by an exact sequence,

• If $\mathrm{Ext}^0(E_a, E_{a+1}) \neq 0$ and $\mathrm{Ext}^0(E_a, E_{a+1}) \otimes E_a \mapsto E_{a+1}$ is surjective then $R_{E_{a+1}}(E_a)$ is defined by the exact sequence,

$$0 \mapsto E_a \mapsto \mathrm{Ext}^0(E_a, E_{a+1})^* \otimes E_{a+1} \mapsto R_{E_{a+1}}(E_a) \mapsto 0\,, \tag{7.8}$$

• If $\mathrm{Ext}^0(E_a, E_{a+1}) \neq 0$ and $\mathrm{Ext}^0(E_a, E_{a+1}) \otimes E_a \mapsto E_{a+1}$ is injective then $R_{E_{a+1}}(E_a)$ is defined by the exact sequence,

$$0 \mapsto R_{E_{a+1}}(E_a) \mapsto E_a \mapsto \mathrm{Ext}^0(E_a, E_{a+1})^* \otimes E_{a+1} \mapsto E_{a+1} \mapsto 0\,, \tag{7.9}$$

• If $\mathrm{Ext}^1(E_a, E_{a+1}) \neq 0$ then $L_{E_a}(E_{a+1})$ is defined by exact sequence

$$0 \mapsto \mathrm{Ext}^1(E_a, E_{a+1})^* \otimes E_{a+1} \mapsto R_{E_{a+1}}(E_a) \mapsto E_a \mapsto 0\,. \tag{7.10}$$

As far as the Chern characters are concerned the new sheaves $L_{E_a}(E_{a+1})$ and $R_{E_{a+1}}(E_a)$ are such that

$$\begin{aligned} \pm\mathrm{ch}(L_{E_a}(E_{a+1})) &= \mathrm{ch}(E_{a+1}) - \chi(E_a, E_{a+1})\,\mathrm{ch}(E_a)\,, \\ \pm\mathrm{ch}(R_{E_{a+1}}(E_a)) &= \mathrm{ch}(E_a) - \chi(E_a, E_{a+1})\,\mathrm{ch}(E_{a+1})\,. \end{aligned} \tag{7.11}$$

where this follows from the exact sequences used in the definition of the mutation and $\pm$ depends on which mutation one uses. The left and the right mutations are inverse of each other and satisfy the braid group relations [1][45]

$$\begin{aligned} L_a L_b &= L_b L_a\,,\ R_a R_b = R_b R_a\,, \quad \text{if } |a-b| > 1\,, \\ L_a L_{a+1} L_a &= L_{a+1} L_a L_{a+1}\,,\ R_a R_{a+1} R_a = R_{a+1} R_a R_{a+1}\,. \end{aligned} \tag{7.12}$$

These transformations implement the braid group action on the collection of exceptional sheaves [45].

A helix of period n, $\{E_i \mid i \in \mathbf{Z}\}$ is a collection of infinitely many exceptional sheaves such that [16, 45]

$$\{E_{i+1}, \quad \cdots \quad , E_{i+n}\}\,, \text{ is an exceptional collection for all } i \in \mathbf{Z}, \tag{7.13}$$

$$E_{i+n} \quad = \quad R_{E_{i+n-1}} \cdots R_{E_{i+2}} R_{E_{i+1}}(E_i)\,. \tag{7.14}$$

[1] $L_a \equiv L_{\mathcal{E}_a}, R_a \equiv R_{\mathcal{E}_a}$.

Thus given any exceptional collection $\{E_1, \cdots, E_n\}$ we can define a helix by extending the exceptional collection periodically i.e., $E_{i+n} = R_{E_{i+n-1}} \cdots R_{E_{i+1}}(E_i)$ and $E_{-i+n} = L_{E_{-i+n-1}} \cdots L_{E_{i-1}}(E_i)$ for $0 \le i \le n$. The exceptional collection defining a helix is called the foundation of a helix. Such an exceptional collection generates the derived category of X [45]. For any exceptional collection $\{E_i \mid i = 1, \cdots, n\}$ which is the foundation of a helix [45]

$$R_{E_{i+n}} \cdots R_{E_{i+1}}(E_i) = E_i \otimes \omega_X \,, \tag{7.15}$$

where ω_X is the canonical line bundle of X. The collection of line bundles $\{\mathcal{O}(0), \mathcal{O}(1), \cdots, \mathcal{O}(n)\}$ on $\mathbb{P}^n$ provides an important example of an exceptional collection which is also the foundation of a helix of period $n+1$. Their Chern character is given by $\mathrm{ch}(\mathcal{O}(k)) = e^{kx}$ where $\int_{\mathbb{P}^n} x^n = 1$. In this case [16]

$$R_{\mathcal{O}(n)} \cdots R_{\mathcal{O}(1)}(\mathcal{O}(0)) = \mathcal{O}(0) \otimes \omega_{\mathbb{P}^n} = \mathcal{O}(n+1). \tag{7.16}$$

The bilinear form for the exceptional collection on $\mathbb{P}^n$ is given by

$$\chi(\mathcal{O}(a), \mathcal{O}(b)) = \binom{n+a-b}{a-b}, \quad a \le b \tag{7.17}$$

$$= 0 \,, \qquad a > b \,. \tag{7.18}$$

If we consider D-branes corresponding to exceptional sheaves on the Kähler manifold we can couple them to sigma models. In this context, as discussed earlier in this paper (3.46), $\mathrm{Ext}^i(E,F)$ is interpreted as the ground states in the open string sector stretched between E and F with fermion number i.

The similarities of the objects defined above and the D-branes we have studied in the context of LG models is striking and as we will discuss below not accidental: The D-branes we have constructed in the LG model turn out to be the mirror of the exceptional bundles. In particular the property that $\dim \mathrm{Ext}^i(E,E) = \delta_{i,0}$ is the statement we discussed before, namely the open string sector of a string stretched between the same D-brane has only one vacuum with fermion number 0 (the function W gives a Morse function on it with exactly one critical point corresponding to an absolute minimum). Also the system of exceptional collection of sheaves has a natural parallel in the LG system: If we consider the D-branes on the cycles γ_i that we constructed, ordered with decreasing value of $\mathrm{Im}W$, then the fact that for $i < j$ the open string stretched between γ_i and γ_j has no Ramond ground states and that for $i > j$ there can only be zero modes in this sector at a fixed fermion number, as discussed before, is exactly the conditions imposed on a collection of exceptional sheaves. Moreover the braiding with left and right mutations has also a natural parallel: If we change the combination of left and right supercharges that we are preserving the image of

the γ_i in the W-plane will rotate by the corresponding angle. Moreover as we change the angle two neighboring γ_i and γ_{i+1} might switch order. In this case the switched γ_i define a different basis for $H_n(\mathbf{C}^n, \mathrm{Re}We^{-i\theta} > 0)$ related by Picard-Lefshetz action as discussed before. This is exactly the same as the change in the Chern characters of transmuted exceptional sheaves, up to the $\pm$ sign, which one can interpret as the orientation of the corresponding LG D-brane. Moreover the left versus right mutation corresponds to the reversal of the direction of change of θ. Finally if we consider a massive sigma model with N isolated vacua, taking any γ_i around in the W plane, as discussed before, is equivalent to changing the $\theta_i k_i$ B-fields of the sigma model by c_1 of the manifolds, which on the D-brane is realized as a tensoring with a $U(1)$ connection with curvature given by c_1, i.e. tensoring the D-brane with ω_X. This is exactly the condition of having a helix of period N. Moreover once we discuss why γ_i are the mirrors of the corresponding D-branes it becomes clear why, up to braidings, the number of solitons in Fano varieties (with only massive vacua) are given by the index of the $\overline{\partial}$ operator coupled to $E_i^* \otimes E_j$ where E_i, E_j belong to a collection of exceptional bundles on the Fano variety.

Below we will present many examples of this connection. The discussions are aimed at giving a sample rather than an exhaustive search through examples.

7.3 $\mathbb{P}^n$

The $\mathbb{P}^n$ sigma model is realized as the $U(1)$ gauge theory with $N = n+1$ matters of charge 1. The mirror is the A_n affine Toda field theory with the superpotential

$$W = \mathrm{e}^{-Y_1} + \cdots + \mathrm{e}^{-Y_n} + \lambda\, \mathrm{e}^{Y_1 + \cdots + Y_n}, \tag{7.19}$$

where $\lambda = \mathrm{e}^{-t} = \mathrm{e}^{-r+i\theta}$. There are $n+1$ critical points $p_0, \ldots, p_n$ given by $\mathrm{e}^{-Y_i}(p_k) = \lambda^{\frac{1}{n+1}} \mathrm{e}^{\frac{2\pi k i}{n+1}}$ with the critical value $w_k = (n+1)\lambda^{\frac{1}{n+1}} \mathrm{e}^{\frac{2\pi k i}{n+1}}$.

As we saw in section 6, the mirror of the trivial bundle $\mathcal{O}(0)$ for the sigma model with $\theta = 0$ is the middle dimensional cycle whose image in the W-plane is a straight line starting at the critical value w_0 and extending along the real axis. We have also seen that the mirror of the bundle $\mathcal{O}(k)$ for the sigma model with $\theta = 0$ is the Lagrangian submanifold whose image in the W-plane is a straight line emanating from w_{-k} in the radial direction. This has been obtained by the shift $\theta = 0 \to \theta = -2\pi k$, i.e. the rotation $\lambda^{\frac{1}{n+1}} \to \lambda^{\frac{1}{n+1}} \mathrm{e}^{-\frac{2\pi k i}{n+1}}$, and interpreting the result as the $\mathcal{O}(k)$ bundle for the sigma model with $\theta = 0$. In this case, the unbroken supercharge is of the A-type combination[1]

[1]Note that we have switched back to the standard convention of chirality: Y_i's are *chiral* superfield rather than twisted chiral superfield. That is why the unbroken supercharges are A-type rather than

$Q = \overline{Q}_+ + \mathrm{e}^{-\frac{2\pi k i}{n+1}} Q_-$ and its conjugate $Q^\dagger$. At this stage, one can rotate the Lagrangian cycle (without touching θ) so that the image in the W-plane is parallel to the real axis. In such a case, the corresponding D-brane preserves the standard combination $Q = \overline{Q}_+ + Q_-$ and its conjugate $Q^\dagger$.

These are depicted in the example of $\mathbb{P}^5$ in Figure 19. For the bundles $\mathcal{O}(2)$, $\mathcal{O}(3)$,...

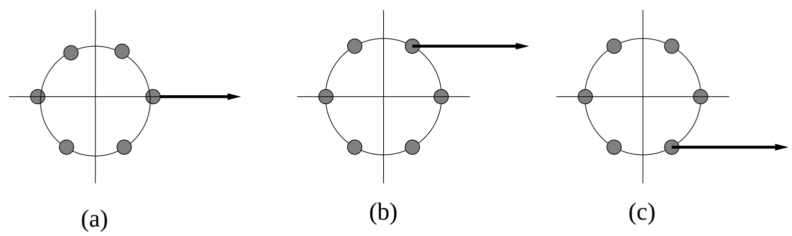

Figure 19: a) $\{\mathcal{O}(0), \theta = 0\} = \mathcal{O}(0)$, b) $\{\mathcal{O}(0), \theta = 2\pi\} = \mathcal{O}(-1)$, c) $\{\mathcal{O}(0), \theta = -2\pi\} = \mathcal{O}(1)$

or $\mathcal{O}(-2)$, $\mathcal{O}(-3)$,..., it is impossible for rotating the cycle so that the images in the W-plane are parallel to the real axis without passing through other critical values. One can avoid this cross-over by bending the branes although it results in the breaking of the supersymmetry. Bending in the clockwise direction, we obtain the collection of bundles $\{\mathcal{O}(0), \cdots, \mathcal{O}(n+1)\}$ which are exceptional collections as we have seen above. By partially changing the direction of bending, we can obtain other exceptional collections as shown in Fig. 20 for the case of $\mathbb{P}^5$. If we order the lines in terms of decreasing asymptotic imaginary part then the exceptional collection in Fig. 20(b) is $\{\mathcal{O}(-3), \mathcal{O}(-2), \mathcal{O}(-1), \mathcal{O}(0), \mathcal{O}(1), \mathcal{O}(2)\}$ and the exceptional collection in Fig. 20(c) is $\{\mathcal{O}(-2), \mathcal{O}(-1), \mathcal{O}(0), \mathcal{O}(1), \mathcal{O}(2), \mathcal{O}(3)\}$.

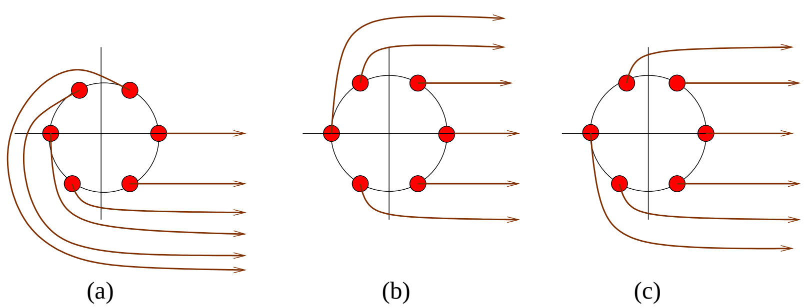

Figure 20: Three exceptional collections which are related to each other by mutations.

B-type.

The observation of [46, 16] can now be understood as a consequence of mirror symmetry. The soliton numbers between different vacua are given by intersection number of middle dimensional cycles determined by the superpotential as shown in section 2. Mirror symmetry relates these cycles and their intersection form to bundles on $\mathbb{P}^n$ and the bilinear form $\chi(\mathcal{E}, \mathcal{F})$ respectively. To find the soliton numbers from this data, as reviewed in section 2, we need to choose suitable classes of cycles. This configuration of cycles is related to the D-branes we have by some Picard-Lefshetz action, which is the mirror realization of Left/Right mutations discussed in the case of exceptional bundles. Let us first discuss LG analog of mutation and then return to the computation of soliton numbers using the $\chi(\mathcal{E}, \mathcal{F})$.

Let us denote the D-brane corresponding to the $-i$-th critical point by C_i. Note that from the mirror symmetry map we have for $i > j$, $C_j \circ C_i = \chi(\mathcal{O}(j), \mathcal{O}(i)) = (n+i-j)!/n!(i-j)!$. Consider as an example the case of $\mathbb{P}^2$ and the bundle $\mathcal{O}(2)$ shown in Fig. 21. Making the middle dimensional C_2 cycle pass through the the critical value w_2 and using the Picard Lefshetz formula

$$C_2' = C_2 - (C_1 \circ C_2) C_1 = C_2 - 3\, C_1 \, . \tag{7.20}$$

which is exactly the same as how the left mutation acts on Chern character upon left mutation of the $\mathcal{O}(2)$ bundle over $\mathcal{O}(1)$. We thus identify C_2' as the mirror of the $\mathbf{L_1}\mathcal{O}(2)$ (with the opposite orientation). Moreover in order for charges not to change, we see that we create three new D-branes $+3C_1$. If we again make the cycle C_2' pass through w_0 we see using the Picard Lefshetz formula that,

$$C_2'' = C_2' - (C_0 \circ C_2') C_0 = C_2 - 3\, C_1 + 3\, C_0 \, . \tag{7.21}$$

Thus we see that we have created $-3C_0$ branes, for charge conservation. Again we see

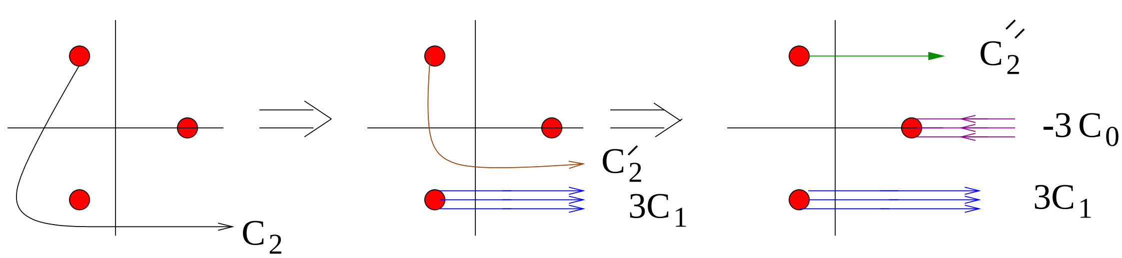

Figure 21: The Picard-Lefshetz Monodromy, leading to brane creation, can be interpreted in terms of left mutations on the mirror side

that we can identify C_2'' with the mirror of $\mathbf{L_0 L_1}(\mathcal{O}(2))$. In other words the above process

viewed in terms of the mutation of $\mathcal{O}(2)$ in the exceptional collection $\{\mathcal{O}(0), \mathcal{O}(1), \mathcal{O}(2)\}$, is given as

$$
\begin{aligned}
\{\mathcal{O}(0), \mathcal{O}(1), \mathcal{O}(\mathbf{2})\} &\mapsto \{\mathcal{O}(0), \mathbf{L_1}\mathcal{O}(\mathbf{2}), \mathcal{O}(1)\} \mapsto \{\mathbf{L_0 L_1}\mathcal{O}(\mathbf{2}), \mathcal{O}(0), \mathcal{O}(1)\} \\
\mathrm{ch}(L_1\mathcal{O}(2)) &= \mathrm{ch}(\mathcal{O}(2)) - 3\,\mathrm{ch}(\mathcal{O}(1))\,, \qquad (7.22)\\
\mathrm{ch}(L_0 L_1\mathcal{O}(2)) &= \mathrm{ch}(L_0\mathcal{O}(2)) - 3\,\mathrm{ch}(L_0\mathcal{O}(1)) \\
&= \mathrm{ch}(\mathcal{O}(2)) - 3\,\mathrm{ch}(\mathcal{O}(1)) + 3\mathrm{ch}(\mathcal{O}(0)) = \mathrm{ch}(\mathcal{O}(-1))
\end{aligned}
$$

The fact that mutating $\mathcal{O}(2)$ through all the other critical points gives $\mathcal{O}(-1) = \mathcal{O}(2) \otimes \mathcal{O}(-3)$ and that $\mathcal{O}(-3)$ is the inverse of c_1 of the canonical bundles, is related to the axial anomaly of the $\mathbb{P}^2$ sigma model.

The left mutated bundle $L_1(\mathcal{O}(2))$ is shown in the figure below. Fig. 22(a) and

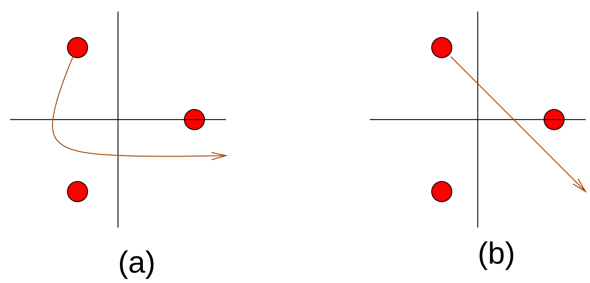

Figure 22: The Branes depicted here are mirrors of $L_1(\mathcal{O}(2))$

Fig. 22(b) represent two different representatives of the homology class mirror to $L_1(\mathcal{O}(2))$. In the case of Fig. 22(a) the representative is not a supersymmetric cycle since its image in the W-plane is not a straight line. The representative shown in Fig. 22(b), however, is supersymmetric and preserves A-model supercharge $\bar{Q}_+ + e^{i\alpha}Q_-$, where α is the angle that the straight line makes with the real axis. This comment also applies to the branes depicted in figure 19, and other branes that will be discussed in this section.

As another interesting example of mutation consider the right mutation of the pair $\{\mathcal{O}(0), \mathcal{O}(1)\}$ on $\mathbb{P}^n$ as shown in Fig. 23. Since $\mathrm{Ext}^0(\mathcal{O}(0), \mathcal{O}(1)) = H^0(\mathcal{O}(0), \mathcal{O}(1)) \neq 0$, we can use the Euler sequence [16]

$$0 \mapsto \mathcal{O}(0) \mapsto H^0(\mathcal{O}(0), \mathcal{O}(1))^* \otimes \mathcal{O}(1) \mapsto T \mapsto 0 \qquad (7.23)$$

In fact $R_{\mathcal{O}(1)}(\mathcal{O}(0))$ is T, the tangent bundle, and its mirror is identified in the LG theory with the brane ${\gamma_0}'$, shown in Fig. 23(b), which is obtained by Picard-Lefshetz monodromy

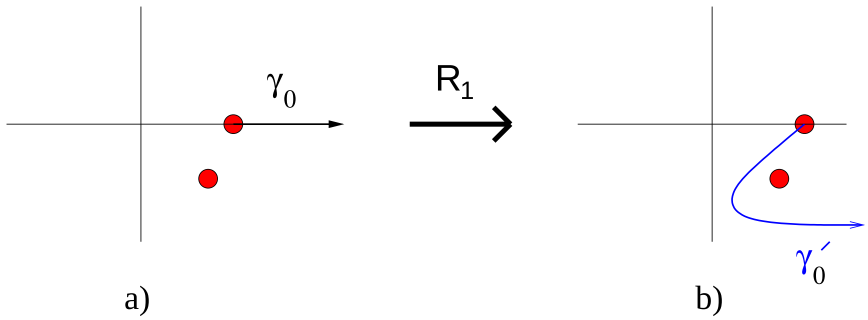

Figure 23: The mirror of the tangent bundle is the D-brane shown in (b).

action on γ_0.[2]

Now we come back to the question of counting the soliton numbers. The soliton number between w_0 and w_{-k} can be computed by first left mutating the $\mathcal{O}(k)$ brane through $\mathcal{O}(k-1), ..., \mathcal{O}(1)$ and then taking its inner product with $\mathcal{O}(0)$. Since $\mathrm{ch}(L_1 \cdots L_{k-1}\mathcal{O}(k)) = \sum_{i=0}^{k-1}(-1)^i\binom{n+1}{i}\mathrm{ch}(\mathcal{O}(k-i))$, hence soliton numbers between two different vacua w_0 and w_{-k} is equal to

$$\begin{aligned} \mu_{i,i-k} = \mu_{0,-k} &= \chi(\mathcal{O}(0), L_1 \cdots L_{k-1}\mathcal{O}(k)) \\ &= \sum_{i=0}^{k-1}(-1)^i \binom{n+1}{i}\binom{n+k-i}{k-i} = (-1)^{k-1}\binom{n+1}{k} \end{aligned} \tag{7.24}$$

in agreement with what we had obtained before for the soliton numbers.

7.4 Toric del Pezzo Surfaces

We next consider the non-linear sigma model with two dimensional toric Fano varieties as the target space. We will show that the supersymmetric cycles of the mirror LG theory, which are the preimages of the straight lines in the W-plane defined by the superpotential, are related to an exceptional collection of bundles on the target space.

From the classification toric Fano varieties it is known that there are five toric Fano surfaces. The toric diagram of these surfaces is captured by a dual lattice shown in Fig. 24 (see [9] and [47] for a detailed discussion) which is obtained naturally from the mirror symmetry description derived in [2]. The first four diagram are that of $\mathbb{P}^2$ and its three blow ups respectively. Fig. 24(e) is the toric diagram of $F_0 = \mathbb{P}^1 \times \mathbb{P}^1$.

[2] It is natural to conjecture that the mirror of all bundles on $\mathbb{P}^n$ is given on the LG mirror by the D-branes corresponding to exceptional bundles with multiplicities given by the decomposition of its Chern character.

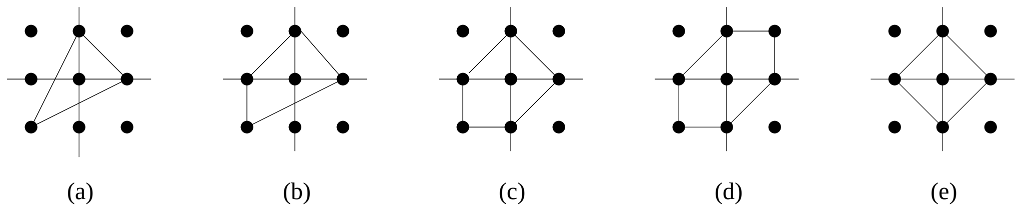

Figure 24: Toric diagram for $\mathbb{P}^2$, its three blow ups and $\mathbb{P}^1 \times \mathbb{P}^1$.

The superpotential of the mirror LG theory can be written from the toric data given in the figure. Let $\{v^{(a)} = (v_1^{(a)}, v_2^{(a)}) \mid a = 1, \cdots, N\}$ be the set of vertices. Then [2]

$$W(X) = \sum_{a=1}^{N} C_a X_1^{v_1^{(a)}} X_2^{v_2^{(a)}} , \tag{7.25}$$

where C_a are complex numbers. Only a subset of C_a are actually physical since some of them can be absorbed by rescaling X_i.

7.4.1 $\mathbb{P}^2$

From the discussion in the previous section we know that the lines shown in the Fig. 25(a) correspond to the exceptional collection $\{\mathcal{O}(-1), \mathcal{O}(0), \mathcal{O}(1)\}$. The exceptional collection shown in Fig. 25(b) is $\{\mathcal{O}(-1), V, \mathcal{O}(0)\}$ where $V = L_0\mathcal{O}(1)$ is such that $\mathrm{ch}(V) = (-2, 1, \frac{1}{2})$. Here we are using the notation such that $\mathrm{ch}(V) = (c_0(V), c_1(V), \int_X \mathrm{ch}_2(V))$. What we actually mean by the negative number for c_0 is that if we reverse the orientation of the D-brane we obtain the corresponding mirror of the bundle. In other words the Chern character is multiplied by a minus sign, when comparing with the charges of the LG D-brane. We shall be somewhat implicit about this in this section, but it should be clear from the context what we mean– namely c_0 of the bundle should always be positive. The soliton counting matrix can be determined from the exceptional collection shown in Fig. 25(b) and is given by $\chi(E_i, E_j)$ where $E_i \in \{\mathcal{O}(-1), V, \mathcal{O}(0)\}$,

$$S_{ij} = \chi(E_i, E_j) = \begin{pmatrix} 1 & -3 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} . \tag{7.26}$$

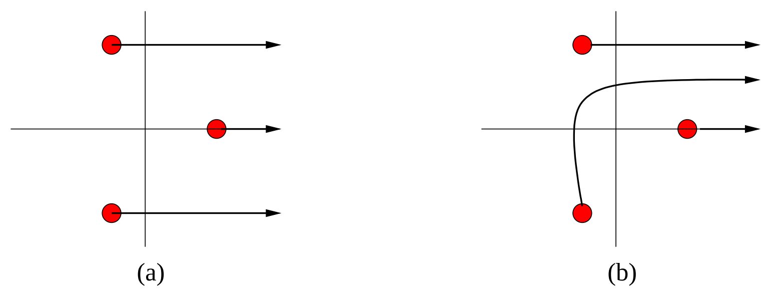

Figure 25: Two exceptional collections related by a left mutation.

7.4.2 $\mathcal{B}_1$

The superpotential is given by

$$W(X_1, X_2) = X_1 + X_2 + \frac{e^{-t}}{X_1 X_2} + \frac{e^{-t_{E_1}}}{X_1}, \tag{7.27}$$

where $t_{E_1} = A(E) - i\,\theta(E)$ is the complexified Kähler parameter of the exceptional cycle, E_1. After rescaling the coordinates we can express the superpotential in the following form which will be more useful for the later discussion,

$$W(X_1, X_2) = e^{-\frac{t}{3}}(X_1 + X_2 + \frac{1}{X_1 X_2} + \frac{e^{-t_{E_1}+\frac{2}{3}t}}{X_1}). \tag{7.28}$$

We define $\mu_1 = e^{-t_{E_1}+\frac{2}{3}t}$, then the critical values of the superpotential given above are

$$w = e^{-\frac{t}{3}}\,(3\,y_* + 2\,\mu_1\,y_*^2)\,, \quad \text{where} \quad \mu_1 y_*^4 + y_*^3 - 1 = 0\,. \tag{7.29}$$

For $|\mu_1| << 1$ we see that

$$y_* \;\approx\; \{e^{2\pi i k/3} + O(\mu_1), -\tfrac{1}{\mu_1} + O(\mu_1) \mid k = 0,1,2\}\,, \tag{7.30}$$

$$w \;\approx\; e^{-\frac{t}{3}}\{3\,e^{2\pi i k/3} + O(\mu_1)\,, -\tfrac{1}{\mu_1} + O(\mu_1) \mid k = 0,1,2\}. \tag{7.31}$$

We will denote the first three critical values as $\{w_k \mid k = 0,1,2\}$ and the fourth one as $\widehat{w}_1$. Thus we see that for $|\mu_1|$ very small the transformation $\mu \mapsto \mu e^{i\theta}$ does not change the three symmetrically located critical values w_k much but the critical value $\widehat{w}_1$ undergoes a clockwise rotation by an angle θ as shown in Fig. 26. Thus to determine the bundles associated with the semi-infinite lines we first consider $\mu_1 \mapsto 0$. The critical value $\widehat{w}_1$ goes to $-\infty$ and we are left with the case of $\mathbb{P}^2$ for which we know the correspondence. Thus the three bundles V_1, V_2 and V_3 shown in Fig. 27 correspond to the pull back of $\mathcal{O}(-1), \mathcal{O}(0)$ and $\mathcal{O}(1)$ from $\mathbb{P}^2$ to $\mathcal{B}_1$ respectively. To determine the bundle associated

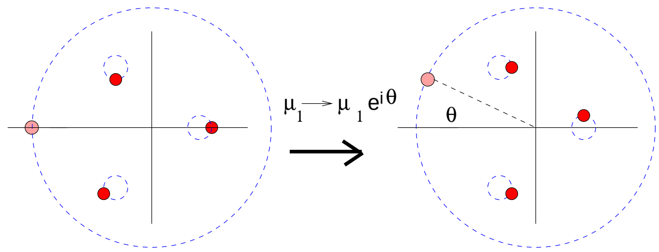

Figure 26: The effect of transformation $\mu_1 \mapsto \mu_1 e^{i\theta}$ on the critical values.

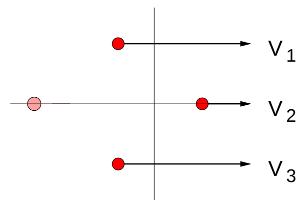

Figure 27: The pull back from $\mathbb{P}^2$ to $\mathcal{B}_1$ of $\{\mathcal{O}(-1), \mathcal{O}(0), \mathcal{O}(1)\}$.

with the critical value $\widehat{w}_1$ we take μ_1 to be very small and positive so that $\widehat{w}_1$ is far from other critical values. In this case the critical value is on the negative real axis as shown in the Fig. 27. Consider the case, as shown in Fig. 28, when there is the D-brane mirror to $\mathcal{O}(0)$, represented by the straight line starting from w_0, present. As mentioned before under the transformation $\mu_1 \mapsto \mu_1 e^{2\pi i}$ the critical value $\widehat{w}_1$ makes a clockwise rotation around the origin, Fig. 28. As it passes through the D-brane associated with the bundle $\mathcal{O}(0)$, due to brane creation effect discussed earlier, it acquires a D-brane charge consistent with the homology class of the cycle associated with this critical point. To determine the bundle corresponding to this new D-brane we use charge conservation. Recall that μ_1 was defined in terms of t_{E_1} and t, the complexified Kähler parameters of $\mathcal{B}_1$. We have kept t fixed in above discussion therefore since the imaginary part of t_{E_1} is minus the B-field integrated over the exceptional curve E_1 the transformation $\mu_1 \mapsto \mu_1 e^{2\pi i}$ corresponds to turning on the B-field through E_1. Thus we interpret Fig. 28(e) as the $\mathcal{O}(0)$ bundle in the B-field background. This implies that if we denote the cohomology class dual to the exceptional curve E_1 as $[E_1]$, the first Chern class of this bundle ($\mathcal{O}(0)$ in the B-field background) is $c_1(\mathcal{O}(0)) - [E_1]$. Denoting by $\widehat{V}_1$ the bundle mirror to the line starting at $\widehat{w}_1$ we get,

$$\mathrm{ch}(\mathcal{O}(0) \oplus \widehat{V}_1) \quad = \quad e^{c_1(\mathcal{O}(0)) - [E_1]}, \tag{7.32}$$

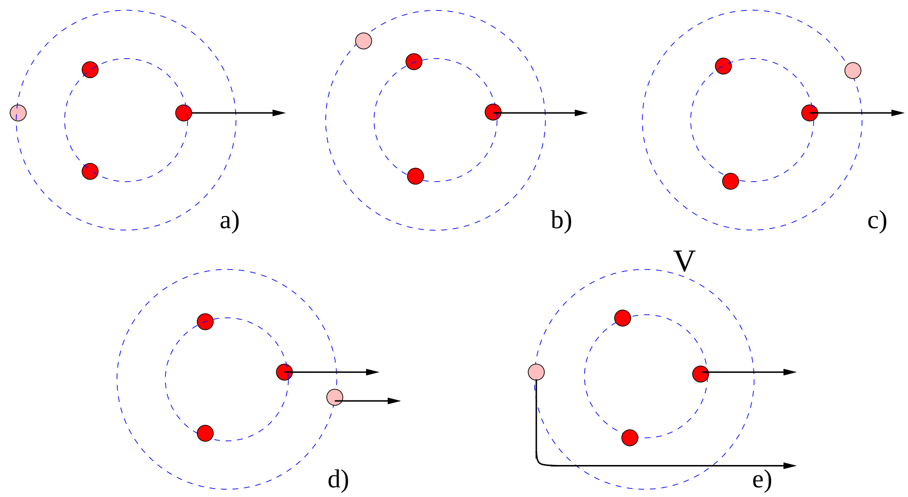

Figure 28: Brane creation as the critical value $\widehat{w}_1$ undergoes a full rotation for $\mu_1 \mapsto \mu_1 e^{2\pi i}$.

$$
\begin{aligned}
\mathrm{ch}(\widehat{V}_1) &= e^{c_1(\mathcal{O}(0)-[E_1]} - \mathrm{ch}(\mathcal{O}(0)) \\
&= (1, [E]_1, -\tfrac{1}{2}) - (1,0,0) = (0, -[E_1], -\tfrac{1}{2})\,.
\end{aligned}
$$

Thus $\widehat{V}_1$ is actually a sheaf with support on the exceptional divisor E_1. The collection of sheaves $\{V_1, V_2, V_3, \widehat{V}_1\}$ is an exceptional collection since $\chi(\widehat{V}_1, \widehat{V}_1) = 1$ and $\chi(\widehat{V}_1, V_i) = 0$ for $i = 1, 2, 3$.

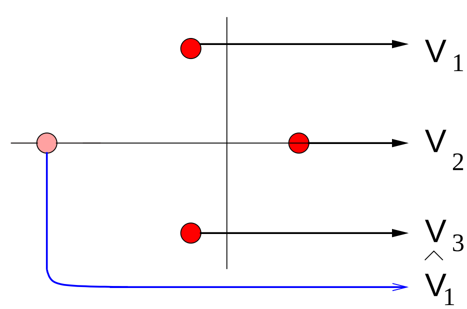

Figure 29: The mirror of an exceptional collection of bundles on $\mathcal{B}_1$.

It is interesting to consider the effect of transformation $e^{-t} \mapsto e^{-t+2\pi i}$. Since $\mu_1 = e^{-t_{E_1}+\frac{2}{3}t}$ we see that under the above transformation $\mu_1 \mapsto \mu_1 e^{\frac{4\pi i}{3}}$. From the previous discussion it would seem that the critical point $\widehat{w}_1$ undergoes a rotation by an angle $\frac{4\pi}{3}$. This, however, is not the case. From eq. (7.28) it is clear that because of the overall factor $e^{-\frac{t}{3}}$ a phase transformation $e^{-t} - \mapsto e^{-t+2\pi i}$ rotates all the critical values by an angle $\frac{2\pi}{3}$.

Thus the critical value $\widehat{w}_1$ gets rotated by $\frac{4\pi}{3} + \frac{2\pi}{3} = 2\pi$,

$$e^{-t} \mapsto e^{-t+2\pi i} : \; w_1 \mapsto w_2 \mapsto w_3 \mapsto w_1 \, , \; \widehat{w}_1 \mapsto \widehat{w}_1 \, . \tag{7.33}$$

The effect on the bundles, however, is more non-trivial and is shown in Fig. 30. V_1', V_2', V_3'

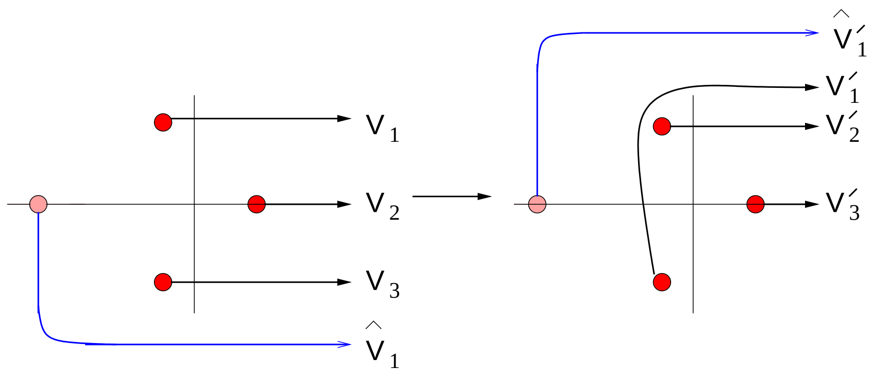

Figure 30: The effect of the transformation $e^{-t} \mapsto e^{-t+2\pi i}$ on the exceptional collection.

and $\widehat{V}_1'$ are V_1, V_2, V_3 and $\widehat{V}_1$ bundles in the background where the B-field through the cycle l has been turned on, where l along with E_1 forms a basis of $H_2(\mathcal{B}_1, \mathbf{Z})$ such that the self-intersection number of l is plus one, $l^2 = 1$. Denoting the cohomology class dual to l by $[l]$ we get

$$\mathrm{ch}(V_k') = e^{c_1(V_k)-[l]} = e^{(k-3)[l]} = (1, (k-3)[l], \tfrac{1}{2}(k-3)^2) \tag{7.34}$$

Thus we see that V_k' is the pull back of $\mathcal{O}(k-3)$ bundle from $\mathbb{P}^2$ to $\mathcal{B}_1$ and can be written as $\{V_1', V_2', V_3'\} = \{V_1 \otimes V_1, V_2 \otimes V_1, V_3 \otimes V_1\}$. The bundle $\widehat{V}_1'$ is easy to determine since under the transformation $t_{E_1} \mapsto t_{E_1} + 2\pi i$ the critical value $\widehat{w}_1$ rotates in the counter clockwise direction therefore argument similar to the one used to determine $\widehat{V}_1$ shows that $\widehat{V}_1'$ is such that

$$\mathrm{ch}(V_2 \oplus \widehat{V}_1') = e^{c_1(V_2)+[E_1]} \implies \mathrm{ch}(\widehat{V}_1') = (0, [E_1], -\tfrac{1}{2}) \, . \tag{7.35}$$

Using this exceptional collection we can calculate the the number of solitons between various vacua. The soliton number between two vacua is given by $\chi(E, F)$ where E and F are the bundle corresponding to the semi-infinite straight lines starting from the two vacua we are studying. The semi-infinite lines must be such that they together do not enclose another critical value. Otherwise the intersection will get contribution from the critical value enclosed by the lines. Thus we first transform to accomplish the exceptional

collection given above into one for which no two lines enclose another critical value. This exceptional collection is shown in Fig. 31 and is obtained from $\{V_1, V_2, V_3, \widehat{V}_1, \}$ by successive left mutations shown in Fig. 31,

$$\begin{aligned} \{V_1, V_2, V_3, \widehat{V}_1\} &\mapsto \{V_1, V_2, L_{V_3}(\widehat{V}_1), V_3\} \\ &\mapsto \{V_1, L_{V_2} L_{V_3}(\widehat{V}_1), V_2, V_3\} \\ &\mapsto \{V_1, L_{V_2} L_{V_3}(\widehat{V}_1), L_{V_2}(V_3), V_2\} =: \{E_1, E_2, E_3, E_4\}\,. \end{aligned} \tag{7.36}$$

As far as the Chern characters of E_i are concerned we have,

$$\mathrm{ch}(E_1) = (1, -1, \tfrac{1}{2})\,,\ \mathrm{ch}(E_2) = (1, -l + [E_1], 0)\,,\ \mathrm{ch}(E_3) = (2, -l, -\tfrac{1}{2})\,,\ \mathrm{ch}(E_4) = (1, 0, 0)\,.$$

The soliton number between the vacua are now given by $\chi(E_i, E_j)$,

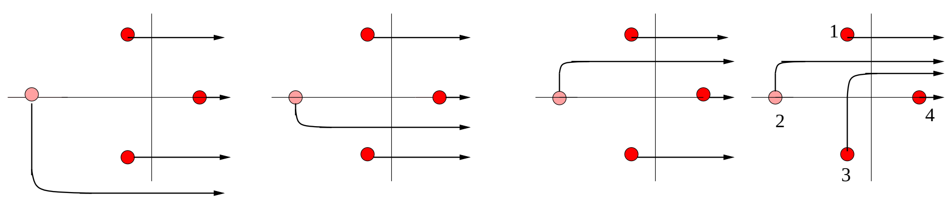

Figure 31: Left mutations of the exceptional collection to obtain an exceptional collection for soliton counting.

$$S_{ij} = \chi(E_i, E_j) = \begin{pmatrix} 1 & -1 & -3 & +3 \\ 0 & 1 & +1 & -2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{7.37}$$

In other words there is one soliton between vacua $2, 1$ and $2, 3$, three solitons between $1, 3$ and $1, 4$ and $3, 4$ and two solitons between vacua $2, 4$. Of course, as we change the Kähler parameters of the manifold the position of the vacua change and the number of solitons change, as reviewed in section 2.

7.4.3 $\mathcal{B}_2$

The superpotential of the LG theory mirror to non-linear sigma model with $\mathcal{B}_2$ is given by

$$W(X_1, X_2) = X_1 + X_2 + \frac{e^{-t}}{X_1 X_2} + \frac{e^{-t_{E_1}}}{X_1} + \frac{e^{-t_{E_2}}}{X_2}\,. \tag{7.38}$$

t_{E_1} and t_{E_2} are the complexified Kähler parameters of the exceptional curves E_1 and E_2 respectively. As in the case of $\mathcal{B}_1$ we rescale the coordinates X_1 and X_2 to obtain the following form for the superpotential which will be useful for later discussion,

$$W(X_1,X_2) = e^{-\frac{t}{3}}(X_1 + X_2 + \frac{1}{X_1 X_2} + \frac{e^{-t_{E_1}+\frac{2}{3}t}}{X_1} + \frac{e^{-t_{E_2}+\frac{2}{3}t}}{X_2})\,. \tag{7.39}$$

We define $\mu_i = e^{-t_{E_i}+\frac{2}{3}t}$ for $i=1,2$. The critical values of the superpotential are given by

$$w = \tfrac{1}{y_*^2-\mu_2} + 2y_* + \mu_1(y_*^2-\mu_2)\,,\ \text{where}\ \ (y_*^2-\mu_2)^2(\mu_1 y_* + 1) - y_* = 0\,. \tag{7.40}$$

For $|\mu_1|, \mu_2| << 1$ we can see that leading order terms for the critical points and critical values are,

$$\begin{aligned}
y_* &= \{e^{\frac{2\pi i k}{3}} + O(\mu_1,\mu_2)\,,\, -\tfrac{1}{\mu_1} + O(\mu_1,\mu_2)\,,\, \mu_2 + O(\mu_1,\mu_2^2) \mid k=0,1,2\}\,,\\
w &= \{3\,e^{\frac{2\pi i k}{3}} + O(\mu_1,\mu_2)\,,\, -\tfrac{1}{\mu_1} + O(\mu_1,\mu_2)\,,\, -\tfrac{1}{\mu_2} + O(\mu_1,\mu_2) \mid k=0,1,2\}\,.
\end{aligned}$$

We will denote the above critical values by $w_0, w_1, w_2, \widehat{w}_1$ and $\widehat{w}_2$ respectively.

To determine the bundle corresponding to $\widehat{w}_1$ we use the same argument as for the case of $\mathcal{B}_1$. As $\mu_1, \mu_2 \mapsto 0$ we recover the $\mathbb{P}^2$ configuration and therefore the the three bundles corresponding to the semi-infinite lines starting at w_1, w_2 and w_3 are the pull backs of the $\mathcal{O}(-1), \mathcal{O}(0)$ and $\mathcal{O}(1)$ bundles from $\mathbb{P}^2$ to $\mathcal{B}_2$ respectively. We will continue to denote these bundle as V_1, V_2 and V_3 respectively as before even though they are different bundles than the ones considered in the last section. However, the Chern classes of these bundles are the same as before. Since in the limit $\mu_2 \mapsto 0$ we recover the $\mathcal{B}_1$ configuration therefore the bundle corresponding to the line starting at $\widehat{w}_1$ is the sheaf with support on the exceptional curve E_1. We will denote it, as before, by $\widehat{V}_1$.

As shown in Fig. 32 the critical value $\widehat{w}_2$ rotates in a clockwise direction as $\mu_2 \mapsto \mu_2\, e^{2\pi i}$. In the presence of D-brane corresponding to V_2 such a rotation of $\widehat{w}_2$ creates a D-brane whose image in the W-plane is the line starting at $\widehat{w}_2$ as shown in Fig. 32. We will denote the corresponding bundle by $\widehat{V}_2$. The transformation $\mu_2 \mapsto \mu_2\, e^{2\pi i}$ corresponds to turning on the B-field through the exceptional curve E_2. Denoting the cohomology class dual to E_2 by $[E_2]$, charge conservation implies that

$$\begin{aligned}
\mathrm{ch}(V_2 \oplus \widehat{V}_2) &= e^{c_1(V_2)-[E_2]}\,,\\
\mathrm{ch}(\widehat{V}_2) &= (0, -[E_2], -\tfrac{1}{2})\,.
\end{aligned} \tag{7.41}$$

The collection of bundles $\{V_1, V_2, V_3, \widehat{V}_1, \widehat{V}_2,\}$ is an exceptional collection. As explained before to calculate the soliton numbers we have to mutate this exceptional collection into

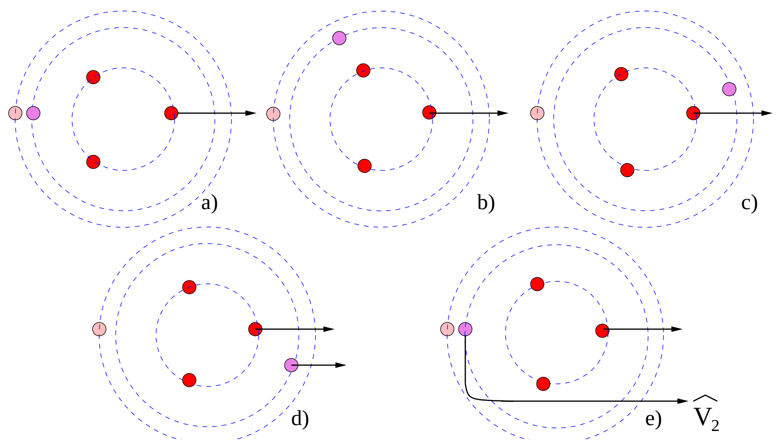

Figure 32: Brane creation as the critical value $\widehat{w}_2$ undergoes a rotation for $\mu_2 \mapsto \mu_2 e^{2\pi i}$.

an exceptional collections for which the corresponding semi-infinite lines are such that any two of them do not enclose a critical point. To obtain such an exceptional collection we consider the right mutations shown in Fig. 33.

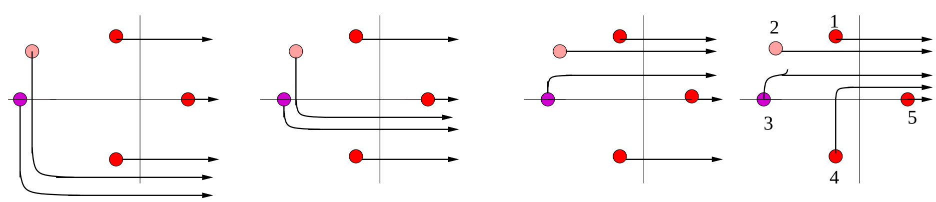

Figure 33: Left mutations of the exceptional collection on $\mathcal{B}_2$ to obtain an exceptional collection for soliton counting.

$$\begin{aligned}
\{V_1, V_2, V_3, \widehat{V}_1, \widehat{V}_2\} &\mapsto \{V_1, V_2, L_{V_3}(\widehat{V}_1), L_{V_3}(\widehat{V}_2), V_3\} \\
&\mapsto \{V_1, L_{V_2}L_{V_3}(\widehat{V}_1), L_{V_2}L_{V_3}(\widehat{V}_2), V_2, V_3\} \\
&\mapsto \{V_1, L_{V_2}L_{V_3}(\widehat{V}_1), L_{V_2}L_{V_2}(\widehat{V}_2), L_{V_2}(V_3), V_2\} =: \{E_1, E_2, E_3, E_4, E_5\}\,.
\end{aligned} \tag{7.42}$$

The soliton numbers between different vacua are now given by $\chi(E_i, E_j)$ where the vacua are labeled in the counter clockwise direction as shown in Fig. 33,

$$S_{ij} = \chi(E_i, E_j) = \begin{pmatrix} 1 & -1 & -1 & -3 & +3 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\,. \tag{7.43}$$

We can read off the number of solitons from the above matrix. Note that the matrix S will change by a braid transformation under the change of parameters. Thus the matrix S given above is the soliton counting matrix as long as the convexity of the critical values shown in Fig. 33 is maintained.

7.4.4 $\mathcal{B}_3$

Now we will consider the case of $\mathbb{P}^2$ blown up at three points, $\mathcal{B}_3$. The superpotential of the mirror LG theory is

$$W_{\mathcal{B}_3}(X_1, X_2) = X_1 + X_2 + \frac{e^{-t}}{X_1 X_2} + \frac{e^{-t_{E_1}}}{X_1} + \frac{e^{-t_{E_2}}}{X_2} + e^{-t_{E_3}} X_1 X_2 \,. \tag{7.44}$$

t_{E_i} are the complexified Kähler parameters of the exceptional curves E_i. After rescaling the coordinates we can write the above superpotential as,

$$W_{\mathcal{B}_3}(X_1, X_2) = e^{-\frac{t}{3}}(X_1 + X_2 + \frac{1}{X_1 X_2} + \frac{e^{-t_{E_1}+\frac{2}{3}t}}{X_1} + \frac{e^{-t_{E_2}+\frac{2}{3}t}}{X_2} + e^{-t_{E_3}-\frac{1}{3}t} X_1 X_2) \,, \tag{7.45}$$

We denote by μ_1, μ_2 and μ_3 the three parameters $e^{-t_{E_1}+\frac{2}{3}t}, e^{-t_{E_2}+\frac{2}{3}t}$ and $e^{-t_{E_3}-\frac{1}{3}t}$ respectively. The positions of the critical values are determined by the μ_i and $e^{-\frac{t}{3}}$ determines the overall scale. To determine the critical values let $\mu_1 = \mu_2 = \mu_3 = \mu$. Then the critical points and the critical values are

$$\begin{aligned} (X_1, X_2) &= \{(e^{\frac{2\pi i k}{3}}, e^{\frac{2\pi i k}{3}}) \,, (\mu^2, -\tfrac{1}{\mu}) \mid k = 0,1,2\} \,, \\ W_{\mathcal{B}_3}(X_1, X_2) &= \{3\, e^{-\frac{t}{3}}(e^{\frac{2\pi i k}{3}} + \mu e^{-\frac{2\pi i k}{3}}), -e^{-\frac{t}{3}}(\tfrac{1}{\mu} + \mu^2) \mid k = 0,1,2\}. \end{aligned} \tag{7.46}$$

The critical value w_4 is degenerate and the three critical values at this point can be separated by taking $\mu_i \neq \mu_j$. For μ close to zero we see that as $\mu \mapsto \mu e^{2\pi i}$ the critical values w_1, w_2, w_3 move in a closed path close to the original critical value. On the other hand the critical value w_4 moves around the origin in the clockwise direction. The solution given above for the critical values in terms of μ is an exact solution. For $\mu_i \neq \mu_j$ we can construct approximate solutions that specify the behavior of critical values for $|\mu_i|$ small. This was done for the case of $\mathcal{B}_1$ and $\mathcal{B}_2$ and the result here is similar to that case. We denote the three degenerate critical values by $\widehat{w}_i$ such that $\mu_i \mapsto 0$ implies $\widehat{w}_i \mapsto -\infty$. Under the transformation $\mu_3 \mapsto \mu_3\, e^{2\pi i}$, $\widehat{w}_3$ undergoes a clockwise rotation around the origin as shown in Fig. 34. Since taking $\mu_1, \mu_2 \mapsto 0$ has no effect on the three symmetrically located critical values therefore we can identify the corresponding bundles as the pull backs of $\mathcal{O}(-1), \mathcal{O}(0)$ and $\mathcal{O}(1)$ from $\mathbb{P}^2$ to $\mathcal{B}_3$, we will denote these bundles

by V_1, V_2 and V_3 respectively. Similarly from Fig. 34 and charge conservation we see that the bundle associated with $\widehat{w}_i$ denoted by $\widehat{V}_i$ is such that,

$$\mathrm{ch}(\widehat{V}_3) = (0, -E_3, -\frac{1}{2})\,. \tag{7.47}$$

It is easy to check that $\{V_1, V_2, V_3, \widehat{V}_1, \widehat{V}_2, \widehat{V}_3\}$ is an exceptional collection. The semi-

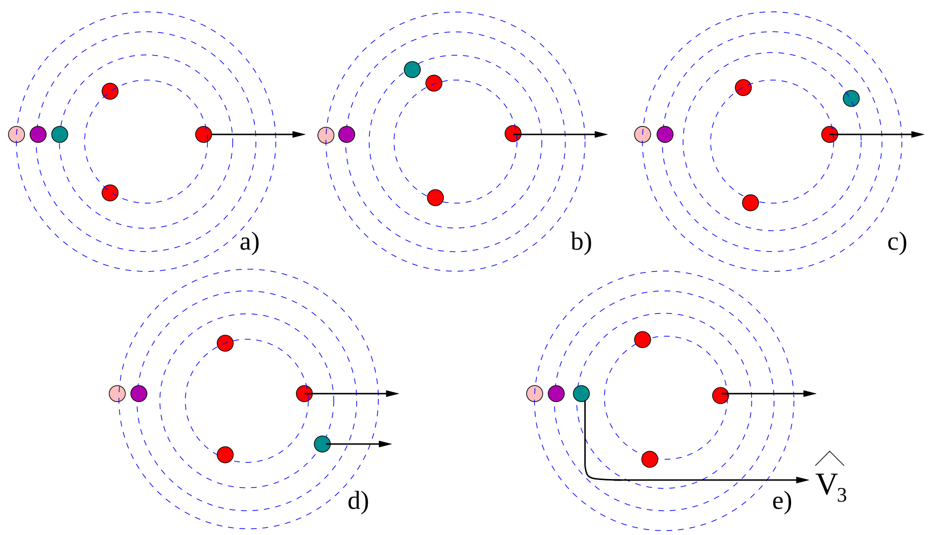

Figure 34: Brane creation as the critical value $\widehat{w}_3$ undergoes a rotation for $\mu_3 \mapsto \mu_3 e^{2\pi i}$.

infinite lines corresponding to this collection are shown in Fig. 35.

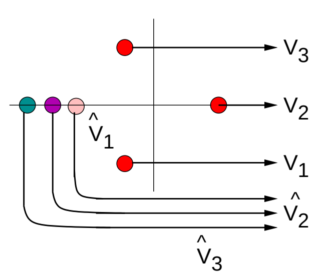

Figure 35: The mirror of the exceptional collection $\{V_1, V_2, V_3, \widehat{V}_1, \widehat{V}_2, \widehat{V}_3\}$.

To calculate the number of solitons between the vacua we need to transform this collection into the one shown in Fig. 36. We see that the exceptional collection shown in Fig. 35 is obtained from $\{V_1, V_2, V_3, \widehat{V}_1, \widehat{V}_2, \widehat{V}_3\}$ by successive left mutations. Note that

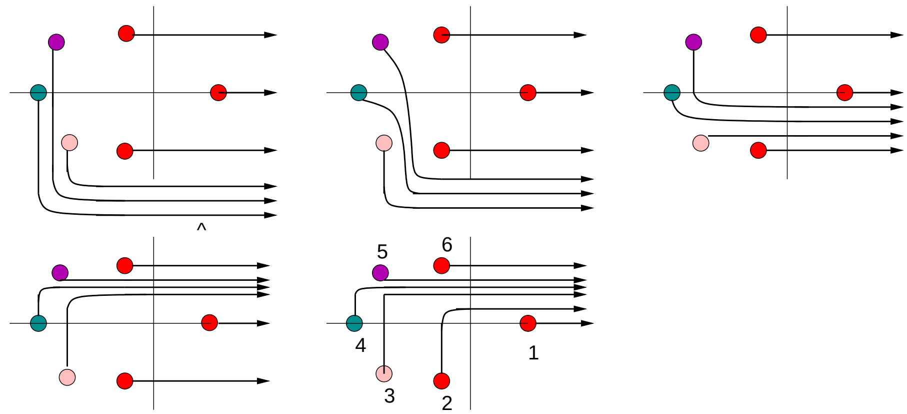

Figure 36: Left mutations of the exceptional collection (shown in Fig. 35) to obtain an exceptional collection for soliton counting.

since $\chi(\widehat{V}_i, \widehat{V}_j) = \delta_{ij}$ therefore we can move the corresponding branes through each other without generating any new branes,

$$
\begin{aligned}
\{V_1, V_2, V_3, \widehat{V}_1, \widehat{V}_2, \widehat{V}_3\} \;&\mapsto\; \{V_1, V_2, V_3, \widehat{V}_2, \widehat{V}_3, \widehat{V}_1\} \\
&\mapsto\; \{V_1, V_2, L_{V_3}(\widehat{V}_2), L_{V_3}(\widehat{V}_3), L_{V_3}(\widehat{V}_1), V_3\} \\
&\mapsto\; \{V_1, L_{V_2}L_{V_3}(\widehat{V}_2), L_{V_2}L_{V_3}(\widehat{V}_3), L_{V_2}L_{V_3}(\widehat{V}_1), V_2, V_3\} \\
&\mapsto\; \{V_2, L_{V_2}L_{V_3}(\widehat{V}_2), L_{V_2}L_{V_3}(\widehat{V}_3), L_{V_2}L_{V_3}(\widehat{V}_1), L_{V_2}(V_3), V_2\} \\
&=:\; \{E_1, E_2, E_3, E_4, E_5, E_6\}\,.
\end{aligned}
\tag{7.48}
$$

The soliton counting matrix is given by

$$
S_{ij} = \chi(E_i, E_j) = \begin{pmatrix} 1 & -1 & -1 & -1 & -3 & +3 \\ 0 & 1 & 0 & 0 & +1 & -2 \\ 0 & 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}
\tag{7.49}
$$

Note that this matrix gives the number of solitons only for those values of the parameters for which we can have the convex configuration shown in Fig. 36.

7.5 $F_0 = \mathbb{P}^1 \times \mathbb{P}^1$

Since F_0 is a product manifold the superpotential of the mirror LG theory is just two copies of the superpotential of LG theory mirror to $\mathbb{P}^1$,

$$W_{F_0}(X_1, X_2) = X_1 + X_2 + \frac{e^{-t_1}}{X_1} + \frac{e^{-t_2}}{X_2}\,.$$

t_1 and t_2 are complexified Kähler parameters of the two $\mathbb{P}^1$'s. After rescaling the variables we can write the above superpotential as

$$W_{F_0}(X_1, X_2) = e^{-\frac{t_1}{2}}(X_1 + \frac{1}{X_1}) + e^{-\frac{t_2}{2}}(X_2 + \frac{1}{X_2})\,. \tag{7.50}$$

The critical points and the corresponding critical values of above superpotential are,

$$\begin{aligned} (X_1, X_2) &= \{(1,1)\,, (1,-1)\,, (-1,1)\,, (-1,-1)\}\,, \\ W(X_1, X_2) &= w_i = \{2e^{-\frac{t_1}{2}} + 2e^{-\frac{t_2}{2}}\,, 2e^{-\frac{t_1}{2}} - 2e^{-\frac{t_2}{2}}\,, -2e^{-\frac{t_1}{2}} + 2e^{-\frac{t_2}{2}}\,, -2e^{-\frac{t_1}{2}} - 2e^{-\frac{t_2}{2}}\}\,. \end{aligned}$$

Without loss of generality we assume that $|e^{-\frac{t_1}{2}}| \geq |e^{-\frac{t_2}{2}}|$. To determine the bundles associated with the cycles $\mathcal{C}_i$ (which are the preimages of the semi-infinite lines in the W-plane) we consider the configuration with $e^{-\frac{t_1}{2}}$ and $e^{-\frac{t_2}{2}}$ real. The critical values in the W-plane in this case lie on the real axis and are non-degenerate as long as $e^{-\frac{t_1}{2}} \neq e^{-\frac{t_2}{2}}$ as shown in Fig. 37. We know that the cycle $\mathcal{C}_1$ (whose image in the W-plane is the

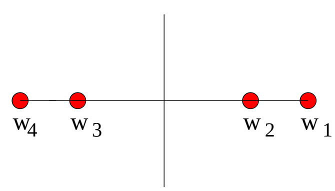

Figure 37: Critical values when $e^{-\frac{t_1}{2}}$ and $e^{-\frac{t_2}{2}}$ are real.

semi-infinite line starting at w_1) is mirror to the trivial bundle $\mathcal{O}(0,0)$ [1]. Note that the transformation $e^{-t_i} \mapsto e^{-t_i}e^{2\pi i}$ has the following effect on the critical values,

$$\begin{aligned} e^{-t_1} \mapsto e^{-t_1}e^{2\pi i} \quad &\Longrightarrow \quad w_1 \leftrightarrow w_3\,,\ w_2 \leftrightarrow w_4\,, \\ e^{-t_2} \mapsto e^{-t_2}e^{2\pi i} \quad &\Longrightarrow \quad w_1 \leftrightarrow w_2\,,\ w_3 \leftrightarrow w_4\,. \end{aligned} \tag{7.51}$$

[1] A bundle $\mathcal{O}(a,b)$ on F_0 is a rank one bundle with first Chern class $c_1 = a\,l_1 + b\,l_2$, where l_1 and l_2 are the generators of $H_2(F_0)$ such that $l_1 \circ l_1 = l_2 \circ l_2 = 0$ and $l_1 \circ l_2 = 1$.

To determine the bundle mirror to the cycle $\mathcal{C}_2$ (whose image in the W-plane is the semi-infinite line starting at w_2 and going to infinity along $e^{i\epsilon}$ ($\epsilon << 1$) we perform the second transformation given above. The effect of this transformation on the image of cycle $\mathcal{C}_1$ in the W-plane is shown in Fig. 38. Thus the bundle,V_2, mirror to $\mathcal{C}_2$ is such that,

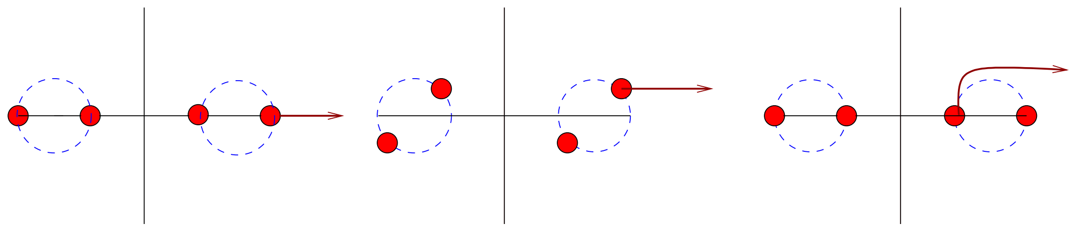

Figure 38: The effect of transformation $e^{-t_2} \mapsto e^{-t_2}e^{2\pi i}$ on the critical values.

$$\mathrm{ch}(V_2) = \mathrm{ch}(\mathcal{O}(0,0)) + \text{B-field} = e^{c_1(\mathcal{O}(0,0))-l_2} = 1 - l_2 = (1, -l_2, 0)\,. \tag{7.52}$$

Thus we can identify V_2 with $\mathcal{O}(0,-1)$. The effect of the first transformation of eq. (7.51) on the cycle $\mathcal{C}_1$ is shown in Fig. 39. Thus by charge conservation we see that the bundle,V_3,

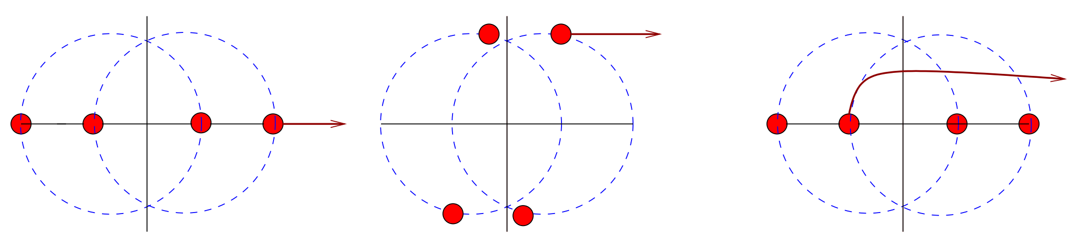

Figure 39: The effect of transformation $e^{-t_1} \mapsto e^{-t_1}e^{2\pi i}$ on the critical values.

mirror to $\mathcal{C}_3$ is such that

$$\mathrm{ch}(V_3) = \mathrm{ch}(\mathcal{O}(0,0)) + \text{B-field} = e^{c_1(\mathcal{O}(0,0))-l_1} = 1 - l_1 = (1, -l_1, 0)\,. \tag{7.53}$$

Thus we can identify V_3 with $\mathcal{O}(-1,0)$. Now consider the transformation

$$(e^{-t_1}, e^{-t_2}) \mapsto (e^{i\theta}e^{-t_1}, e^{i\theta}e^{-t_2})\,,\ \ \theta \in [0, 2\pi]\,. \tag{7.54}$$

The effect of this transformation on the critical values and the cycle $\mathcal{C}_1$ is shown in Fig. 40. Thus charge conservation implies that the bundle V_4 mirror to the cycle $\mathcal{C}_4$ is such that

$$\mathrm{ch}(V_4) = \mathrm{ch}(\mathcal{O}(0,0)) + \text{B-field} = e^{c_1(\mathcal{O}(0,0))-l_1-l_2} = (1, -l_1 - l_2, 1)\,. \tag{7.55}$$

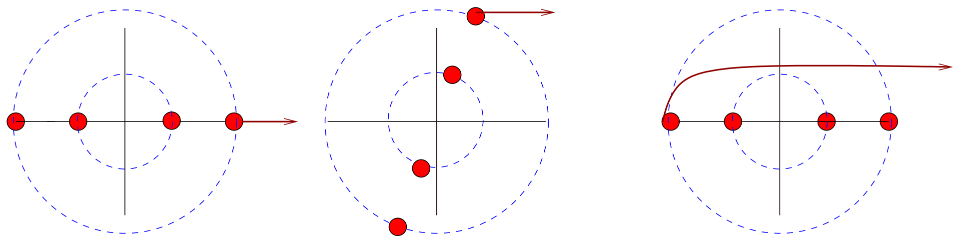

Figure 40: The effect of the combined transformation $(e^{-t_1}, e^{-t_2}) \mapsto e^{2\pi i}(e^{-t_1}, e^{-t_2})$.

Thus we can identify V_4 with $\mathcal{O}(-1,-1)$.

The set of bundles $\{\mathcal{O}(-1,-1), \mathcal{O}(-1,0), \mathcal{O}(0,-1), \mathcal{O}(0,0)\}$ is an exceptional collection and using it we can calculate the soliton counting matrix. But first since we do not want any three vacua to be collinear we deform the configuration using eq. (7.54) for $|\theta| << 1$. And also we need to transform this collection of exceptional bundles into the one shown in Fig. 41 by left or right mutations.

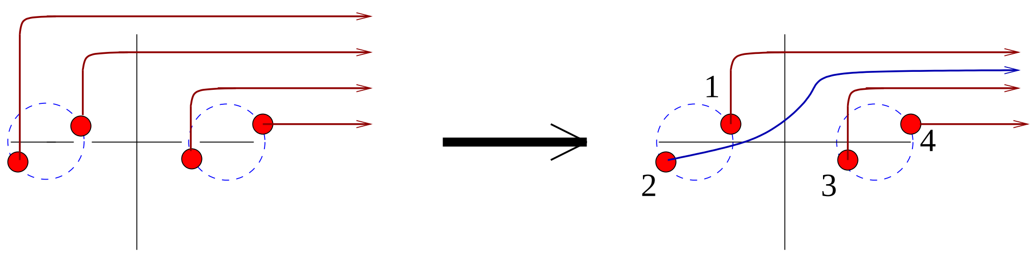

Figure 41: A right mutation of the exceptional collection to obtain another exceptional collection for soliton counting.

$$\begin{aligned}&\{\mathcal{O}(-1,-1)\,,\,\mathcal{O}(-1,0)\,,\,\mathcal{O}(0,-1)\,,\,\mathcal{O}(0,0)\} \mapsto \\ &\qquad \{\mathcal{O}(-1,0)\,,R_{\mathcal{O}(-1,0)}(\mathcal{O}(-1,-1))\,,\,\mathcal{O}(0,-1)\,,\,\mathcal{O}(0,0)\} =: \{E_1\,,\,E_2\,,\,E_3\,,\,E_4\}\,.\end{aligned}$$

The soliton counting matrix S_{ij} is then given by

$$S_{ij} = \chi(E_i, E_j) = \begin{pmatrix} 1 & -2 & 0 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}\,. \tag{7.56}$$

7.6 Higher dimensional toric Fano varieties

As an example of higher dimensional toric Fano varieties we consider the blow ups of projective spaces. Blow up of $\mathbb{P}^{n-1}$ upto n points is a toric Fano variety [48]. Each blow up corresponds to replacing a point by $\mathbb{P}^{n-2}$ and thus each blow up introduces $n-2$ new cohomology elements. We consider the case of maximal blow ups since others can be obtained from this one as we saw for the case of two dimensional del Pezzo surfaces.

The linear sigma model is a $U(1)^n$ gauge theory. The $2n$ chiral superfields have the following charge assignment under the $U(1)^n$ gauge group,

$$\begin{aligned} Q_1 &= (1,1,1\cdots 1,1;\ \ 0,\ \ 0,0,\cdots 0,\ \ 0)\,, \\ Q_2 &= (0,1,1\cdots 1,1;-1,\ \ 0,0,\cdots 0,\ \ 0), \\ Q_3 &= (1,0,1\cdots 1,1;\ \ 0,-1,0,\cdots 0,\ \ 0), \\ &\vdots \\ Q_n &= (\underbrace{1,1,1\cdots 1,0}_{n};\underbrace{\ \ 0,\ \ 0,0,\cdots,0,-1}_{n})\,. \end{aligned} \tag{7.57}$$

The LG superpotential is then given by

$$W(X) = \sum_{i=1}^{n-1} X_i + \frac{e^{-t}}{X_1\cdots X_{n-1}} + \sum_{i=1}^{n-1}\frac{e^{-t_i}}{X_i} + e^{-t_{n-1}}X_1\cdots X_{n-1}\,.$$

After rescaling X_i we can write the above superpotential as

$$W(X) = e^{-\frac{t}{n}}\Big(\sum_{i=1}^{n-1} X_i + \frac{1}{X_1\cdots X_{n-1}} + \sum_{i=1}^{n-1}\frac{e^{-t_i+\frac{2}{n}t}}{X_i} + e^{-t_{n-1}+\frac{n-2}{n}t}X_1\cdots X_{n-1}\Big)\,. \tag{7.58}$$

Let $\mu_i = e^{-t_i+\frac{2}{n}t}$ for $i=1,\cdots,n-2$ and $\mu_{n-1} = e^{-t_{n-1}+\frac{n-2}{n}t}$. Consider the case when $\{\mu_i = \mu \mid i=1,\cdots,n-1\}$, in this case there are $2(n-1)$ critical points given by

$$X_i = f\,,\ (\mu\, f^{n-2}+1)(f^n-1) = 0\,. \tag{7.59}$$

n of these critical points are also the critical point of the $\mathbb{P}^{n-1}$ superpotential. The new critical points and critical values are

$$\begin{aligned} X_i^{(k)} &= \mu^{-\frac{1}{n-2}}e^{\frac{i\pi(2k+1)}{n-2}}\,,\ i=1,\cdots,n-1,\ k=1,\cdots,n-2, \\ \widehat{w}_k &= W(X^{(k)}) = n(\mu^{-\frac{1}{n-2}}e^{\frac{i\pi(2k+1)}{n-2}} + \mu^{n-1}e^{-\frac{i\pi(2k+1)}{n-2}}) \end{aligned} \tag{7.60}$$

For $\mu_i \neq \mu_j$ each of the above new critical points splits up into $n-1$ critical points. Thus the critical value $\widehat{w}_k$ is degenerate with multiplicity $n-1$. If $\mu_i \mapsto 0$ then $n-2$ critical

values go to infinity and the multiplicity of $\widehat{w}_k$ reduces to $n-2$. To determine the bundles corresponding to the lines ending on these new critical values we only need to consider the case when $\mu_1 \neq 0$ and $\mu_i = 0$ for $i = 2, \cdots, n-1$. This is the case of $\mathbb{P}^{n-1}$ blown up at one point. In this case there are $2n-2$ non-degenerate critical points given by the following equation

$$X_i = f\,,\ \mu_1 f^{2n-2} + f^n - 1 = 0\,, \tag{7.61}$$

For μ_1 very small we can write the leading terms in the solution as

$$\begin{aligned} X_i^{(k)} &= f_k = e^{\frac{2\pi i k}{n}} + O(\mu_1)\,,\ X_i^{(k')} = f_{k'} = \mu_1^{-\frac{1}{n-2}} e^{\frac{i\pi(2k'-1)}{n-2}} + O(\mu_1)\,, \\ w_k &= W(X_i^{(k)}) = n\, e^{\frac{2\pi i k}{n}} + O(\mu_1)\,,\ \widehat{w}_{k'} = W(X_i^{(k')}) = \mu_1^{-\frac{1}{n-2}}\, e^{\frac{i\pi(2k'-1)}{n-2}} + O(\mu_1)\,, \end{aligned}$$

where $k \in \{0, \cdots, n-1\}$ and $k' \in \{1, \cdots, n-2\}$. Thus we see that as $\mu_1 \mapsto \mu_1 e^{2\pi i}$ the critical values $\widehat{w}_{k'}$ are rotated by $e^{-\frac{2\pi i}{n-2}}$. Hence $w'_{k'}$ is mapped to $w'_{k'-1}$ by this transformation as shown in Fig. 42 for the case of $\mathbb{P}^5$.

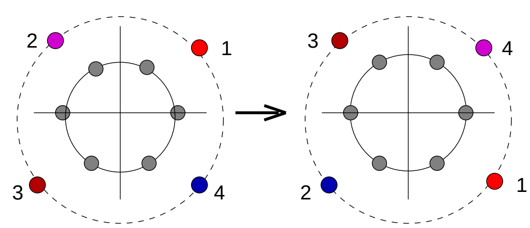

Figure 42: The effect of the transformation $\mu_1 \mapsto \mu_1 e^{2\pi i}$ on the critical values $\widehat{w}_i$ for $\mathbb{P}^5$.

Consider the case of Fig. 43 where we have the D-brane corresponding to the bundle $\mathcal{O}(0)$ ending on the critical value w_0 on the positive real axis. After the transformation $\mu_1 \mapsto \mu_1 e^{2\pi i}$ we create another D-brane whose image in the W-plane is the semi-infinite line starting at w'_{n-2}. Denote by B the cohomology class dual to the exceptional divisor

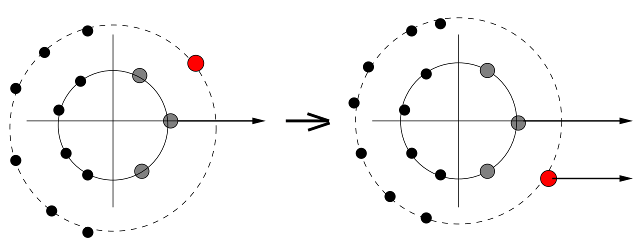

Figure 43: Brane creation as $\widehat{w}_1$ passes through the mirror of the trivial line bundle.

and by V_1 the new bundle created then by charge conservation it follows that,

$$\mathrm{ch}(\mathcal{O}(0) \oplus V) = e^{c_1(\mathcal{O}(0))-B} \Longrightarrow \mathrm{ch}(V) = e^{-B} - 1\,, \tag{7.62}$$

$$c_0(V) = 0\,,\, c_1(V) = -B\,,\, c_i(V) = 0\,,\, i = 2, \cdots, n\,. \tag{7.63}$$

Thus V is the $-[\mathcal{O}(0)]$ bundle on $\mathbb{P}^{n-2}$. A transformation $\mu_1 \mapsto \mu_1\, e^{2\pi i m}$ maps it to $\widehat{w}_{n-2-m}$, thus the bundle which corresponds to the semi-infinite line starting at $\widehat{w}_{k'}$ is the line bundle $\mathcal{O}(n-2-k')$ on $\mathbb{P}^{n-2}$, a sheaf on $\mathbb{P}^{n-1}$ with support on the exceptional divisor. The case of $\mathbb{P}^5$ is shown in Fig. 44.

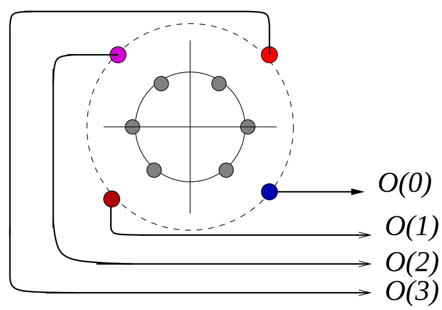

Figure 44: Bundles which are mirror of the lines starting at $\widehat{w}_i$.

8 D-Brane in String Theory and Mirror Symmetry

It is natural to ask how the map between D-branes corresponding to sheaves and the Lagrangian submanifolds in the Landau-Ginzburg model works in the case of conformal theories. In the context of Gepner models, the structure of Cardy states and their sigma model interpretation have been studied [49]. our constructions of Cardy states in terms of Lagrangian submanifolds of LG models lead to a deeper geometric insight in this regard. Since we have considered the case of minimal models in detail, and Gepner model is an orbifold of their tensor products, it is straightforward to identify the relevant D-branes in the orbifold LG model.

There is however, another case of the conformal theory we can consider namely the non-compact CY manifolds. These are the cases of most interest in the context of geometric engineering of QFT's. As an example of this class consider the total space of the canonical line bundle over a compact Fano variety. This space is a non-compact Calabi-Yau manifold. We find relations between the LG theories mirror to the superconformal sigma model on the non-compact Calabi-Yau and the sigma model on the Fano variety.

For simplicity let us assume that the Fano variety is given by the weighted projective space with weights ($q_i > 0$). Then the total space of the canonical bundle is captured by a linear sigma model with a single $U(1)$ gauge theory with matter fields with charges $(-\sum_i q_i, q_1, q_2, ..., q_n)$. The mirror of this is an LG theory [2] with of $n+1$ variables with superpotential,

$$W = \sum_{i=0}^{n} x_i \qquad \text{subject} \quad \text{to} \quad \prod_{i=1}^{n} x_i^{q_i} = e^{-t} x_0^{\sum q_i} \,. \tag{8.1}$$

Recall that the correct field variables are Y_i where $x_i = e^{-Y_i}$. For simplicity let us assume one of the charges say $q_n = 1$ (the more general case can also be done with the additional complication of introducing orbifold groups). Then we can also write the above superpotential as

$$W = x_0[1 + \sum_{i=1}^{n-1} \widehat{x}_i + \frac{e^{-t}}{\prod_{i=1}^{n-1} \widehat{x}_i^{q_i}}] \,. \tag{8.2}$$

Where $\hat{x}_i = e^{-\widehat{Y}_i}$ and $\widehat{Y}_i = Y_i - Y_0$. This change of fields is linear and introduces no Jacobians in the field measure. Now, as far as periods and BPS states which are sensitive only to period integrals $\int \prod d\phi_j e^{-W}$ are concerned this LG theory is equivalent to the LG theory given by

$$W = x_0[1 + \sum_{i=1}^{n-1} \widehat{x}_i + \frac{e^{-t}}{\prod_{i=1}^{n-1} \widehat{x}_i^{q_i}} - uv] \,, \tag{8.3}$$

where now x_0 is the right field variable (i.e. $x_0 \in \mathbf{C}$ rather than $\mathbf{C}^*$), and u, v are chiral fields also taking value in $\mathbf{C}$. To see this note that in the BPS computations integrating over the u, v fields leads to a $\frac{1}{x_0}$ in the measure which combined with dx_0 converts it back to the measure appropriate for x_0 taking values in $\mathbf{C}^*$, and leading to the previous LG periods. Having established their equivalence (at least in the weak sense discussed in [2]), in the period integrals we can integrate out x_0 in the new version of the LG theory, and obtain a $\delta(1 + \sum_{i=1}^{n-1} \widehat{x}_i + \frac{e^{-t}}{\prod_{i=1}^{n-1} \widehat{x}_i^{q_i}} - uv)$. Thus we see that as far as the BPS data is concerned the mirror of the sigma model on the non-compact CY, which is originally the LG model, is equivalent to the sigma model on another non-compact Calabi-Yau given by

$$f(\widehat{x}_i) = uv \qquad \text{where} \qquad f(\widehat{x}_i) = 1 + \sum_{i=1}^{n-1} \widehat{x}_i + \frac{e^{-t}}{\prod_{i=1}^{n-1} \widehat{x}_i^{q_i}} \,, \tag{8.4}$$

where $\hat{x}_i$ take values in $\mathbf{C}^*$ but u, v are variables in $\mathbf{C}$. Note that this non-compact mirror CY has dimension n which is the same as the dimension of the original non-compact CY. The holomorphic n form can be viewed as

$$\Omega = \frac{\prod_{i=1}^{n} \frac{d\widehat{x}_i}{\widehat{x}_i} du dv}{df} = \frac{\prod_{i=1}^{n} \frac{d\widehat{x}_i}{\widehat{x}_i} du}{u} \,. \tag{8.5}$$

It is this version of local mirror symmetry that was first discovered in the literature [50, 51].

We see a striking resemblance between the non-conformal sigma model on the Fano variety and the conformal sigma model on the total space of the canonical line bundle over the Fano variety. In particular the $f(\hat{x}_i)$ appearing in the above formula is precisely the superpotential for the LG theory mirror to the non-conformal sigma model on the Fano variety.

Even though we have presented the above discussion in the context of a linear sigma model with a single $U(1)$, it can be easily generalized to the case with more $U(1)$'s as well as with extra superpotentials corresponding to complete intersections.

BPS states and local mirror symmetry

In computing the BPS states in such local contexts the idea developed in [51] was to consider supersymmetric mid-dimensional cycles on the mirror. Moreover one could simplify the counting of such cycles by considering fibration structure of the non-compact CY and studying the supersymmetric cycles on the fibers and consider the effective tension of the branes as one varies over the base. In this way it was shown in [51], in the context of local mirror of $SU(N)$ gauge theories, how the problem is translated to finding minimal energy string configurations (with varying tensions) on a Riemann surface given by $f(x_1, x_2) = 0$. This was implemented in detail for the $SU(2)$ case where various expected properties of BPS states in the corresponding $\mathcal{N} = 2$ gauge theory in 4 dimensions [52] was recovered including the decay of certain BPS states. Further applications along these lines have been considered [53–56].

We can also connect the above description to the BPS states for the probes in the context of F-theory, which is the subject of the next section.

8.1 Local mirror symmetry and F-theory

In this section we will see that the W-plane geometry of LG theory mirror to sigma model with certain non-compact CY threefolds as target space is closely related to some F-theory backgrounds [57]. The link we find is as follows: The BPS states on the non-compact CY threefold side are D-branes wrapped on compact even dimensional cycles. As discussed above these get transformed on the mirror side to certain 3-cycles in a non-compact CY 3-fold. For the particular backgrounds of interest the non-compact CY 3-fold itself has a simple $\mathbf{C}^*$ fibration structure over CY 2-fold (a local description of elliptic $K3$). The image of the closed 3-cycles get mapped to minimal 2-cycles in this geometry,

which can possibly have boundaries where the $\mathbf{C}^*$ fibration degenerates. This in turn can be viewed as computation of BPS state in a certain F-theory background with a 3-brane probe (placed where the $\mathbf{C}^*$ fibration degenerates).

We will first review the probe theory description for F-theory and its BPS states and then give some examples of non-compact CY manifolds and the corresponding probe theory.

8.1.1 Probe theory and BPS states

Consider a manifold $\mathcal{X}$ which is an elliptic fibration over the complex plane B

$$y^2 \;=\; x^3 + f(z)\,x + g(z)\,, \quad z \in B\,, \tag{8.6}$$

provided with a non-vanishing holomorphic 2-form Ω

$$\Omega \;=\; \lambda\; dz, \tag{8.7}$$

where $\lambda = \frac{dx}{y}$ is the holomorphic 1-form on the elliptic fibers. F-theory compactification on $\mathcal{X}$ is equivalent to type IIB compactification on the base B with a varying coupling constant τ (defined up to $SL(2,\mathbf{Z})$ transformations) given by the complex structure of the elliptic fiber. The position of the degenerate elliptic fibers on the base is given by the zeroes of the discriminant, $\Delta(z)$, of the elliptic fibration (8.6),

$$\Delta(z) = 4\,f(z)^3 + 27\,g(z)^2\,. \tag{8.8}$$

From Picard Lefshetz theory we know that as we go around the position of a degenerate fiber in the base, the complex structure parameter τ undergoes an $SL(2,\mathbf{Z})$ transformation. As mentioned before this complex structure parameter is identified with the coupling constant of type IIB. Since in type IIB monodromies associated with 7-branes transform τ by $SL(2,\mathbf{Z})$ transformations, the position of a degenerate fiber on the base, in type IIB, is associated with a 7-brane. $SL(2,\mathbf{Z})$ symmetry of type IIB then implies the existence of a family of 7-branes labelled by two relatively prime integers, (p,q). As for the case of 7-brane, a (p,q) 7-brane at a point z_* can be associated, in F-theory, with an elliptic fiber over z_*, $T^2_{z_*}$, whose degenerating 1-cycle is $p\alpha + q\beta \in H_1(T^2_{z_*},\mathbf{Z})$.

In ref.[58], $\mathcal{N}$=2 $SU(2)$ Seiberg-Witten theory was interpreted as the worldvolume theory of a D3-brane in the presence of mutually non-local 7-branes. A BPS state of charge (p,q) in the D3-brane theory is a BPS string or a BPS string junction of total asymptotic charge (p,q) with support on 7-branes and ending on the D3-brane. In the

F-theory picture D3-brane lifts to a regular elliptic curve of $\mathcal{X}$. Strings or string junctions stretched between the 7-branes and the D3-brane are, in F-theory, two real dimensional curves in the manifold $\mathcal{X}$ with or without boundary depending on whether the string junction ends on the D3-brane or not[1]. If the string junction ends on a D3-brane the corresponding curve in F-theory has a boundary on the elliptic curve above the position of the D3-brane. The homology cycle of the boundary is determined by the (p,q) charge of the string junction ending on the D3-brane. BPS string junctions correspond to curves holomorphic in the complex structure whose kähler form is Ω. The mass of a BPS state of charge (p,q) is given by the area of the corresponding curve $\mathcal{C}_{p,q}$,

$$M_{p,q} = |\int_{\mathcal{C}_{p,q}} \Omega| \,. \tag{8.9}$$

8.1.2 Superpotentials and F-theory backgrounds

We will see in this section that the non-compact CY 3-folds with an equation of the form $f(x_1,x_2)=uv$ where x_1,x_2 are $\mathbf{C}^*$ variables and u,v are $\mathbf{C}$ variables get related to the F-theory probe description. We first discuss the structure of the BPS D3-branes in the non-compact local Calabi-Yau description and then relate it to the F-theory description.

Instead of being general, we consider a concrete example. The general case is similar. Consider the case of $\mathcal{O}(-3)$ over $\mathbb{P}^2$. This non-compact CY threefold, which is the total space of $\mathcal{O}(-3)$ bundle over $\mathbb{P}^2$, will be denoted by $\mathcal{M}$. This has linear sigma model description in terms of a single $U(1)$ gauge theory with charges of the matter fields $(-3,1,1,1)$. The LG superpotential of the mirror theory is,

$$W(x) = x_0 + x_1 + x_2 + e^{-t}\frac{x_0^3}{x_1x_2} \,. \tag{8.10}$$

As we discussed before as far as the BPS data is concerned the non-compact CY defined by eq. (8.10) is equivalent to another non-compact CY, $\widehat{\mathcal{M}}$, defined by,

$$1 + x_1 + x_2 + \frac{e^{-t}}{x_1x_2} = -z\,, \qquad z = -uv\,. \tag{8.11}$$

where x_1,x_2 are $\mathbf{C}^*$ variables and u,v,z are variables in $\mathbf{C}$. In particular the relevant holomorphic 3-form is given by $\frac{dx_1dx_2du}{x_1x_2u}$ (by eliminating z,v and noting that the denominator has $\partial uv/\partial v$). To better understand the geometry of $\widehat{\mathcal{M}}$ we rewrite the defining

[1]This description follows from the connection between F-theory compactified on a circle and the M-theory in one lower dimension. In the M-theory description the D3-brane probe gets mapped to an M5 brane wrapped over the corresponding elliptic fiber and the BPS states are M2 branes wrapped over 2-cycles of the $K3$ geometry, possibly ending on the M5 brane. The image of the M2 brane projected on the z-plane gives the string junction description in the type IIB setup.

equation eq. (8.11) in the following form,

$$h(x_1,x_2,z) \;:=\; x_1^2x_2 + x_1x_2^2 + 1 + \, z\,x_1x_2 = 0 \tag{8.12}$$

$$z - e^{t/3} \;=\; -uv\,. \tag{8.13}$$

where we have rescaled variables and shifted z. In this form the holomorphic 3-form becomes

$$\Omega = \frac{dx_1dx_2du}{x_1x_2u} = \frac{dx_1dx_2}{\partial h/\partial z}\cdot\frac{du}{u} = \Omega_2\frac{du}{u} \tag{8.14}$$

The first equation in eq. (8.12) defines an elliptic fibration (in terms of the x_1, x_2 variables over the complex plane with coordinate z). Moreover the corresponding two form Ω_2 is the same as would be for a K3 geometry where x_1, x_2 are now viewed as variables in $\mathbf{C}$. It is more convenient to homogenize the above elliptic curve by introducing an extra variable x_0:

$$x_1^2x_2 + x_1x_2^2 + x_0^3 + zx_0x_1x_2 = 0 \tag{8.15}$$

We can convert this into the Weierstrass form by the following coordinate transformation,

$$x_1 = Y + \tfrac{U}{2} + z\tfrac{X}{2}\,,\; x_2 = -Y + \tfrac{U}{2} + z\tfrac{X}{2}\,,\; x_0 = X\,, \tag{8.16}$$

$$UY^2 = X^3 + (\tfrac{z}{2})^2UX^2 + \tfrac{z}{2}U^2X + \tfrac{1}{4}U^3\,, \tag{8.17}$$

where now U is a scaling variable and we can set it to 1. Thus the BPS data of $\widehat{\mathcal{M}}$ is the same as that of

$$Y^2 = X^3 + (\tfrac{z}{2})^2X^2 + \tfrac{z}{2}\,X + \tfrac{1}{4} \qquad,\;\; z - e^{t/3} = -uv\,, \tag{8.18}$$

with the holomorphic 3-form

$$\Omega = \frac{dX}{Y}dz\frac{du}{u}. \tag{8.19}$$

Thus we see that $\widehat{\mathcal{M}}$ can be viewed as the product of an elliptic fibration times a $\mathbf{C}^*$ fibration over z. The elliptic fibration has a discriminant given by $\Delta(z) = \frac{1}{16}(27 - z^3)$, the three degenerate fibers are located symmetrically at $(27)^{\frac{1}{3}}\{1, e^{2\pi i/3}, e^{4\pi i/3}\} =: \{z_1, z_2, z_3\}$. The $\mathbf{C}^*$ fibration has one degenerate fiber given at $z = e^{t/3}$.

To determine supersymmetric 3-cycles in $\widehat{\mathcal{M}}$ we use the circle of $\mathbf{C}^*$ fibration as one cycle times an additional 2-cycle. We note that there are only two closed 2-cycles in this fibration. One is the elliptic fiber of the fibration itself and the other closed 2-cycle is formed by taking a path in the base that encloses z_i and the $(1,0)$ cycle of the elliptic

fiber above this path.[2] The fact that there are no other closed 2-cycles is not obvious even though we do not have degenerating cycles of the same charge, since a cycle starting from a degenerate fiber can undergo monodromy transformations when it goes around other degenerating fibers. One can, however, construct 2-cycles with boundaries such that the boundary is a 1-cycle of an elliptic fiber above the point z_*. We can construct closed 3-cycles using the 1-cycle of the $\mathbb{C}^*$ fibration and the 2-cycles (with boundaries) if we choose the position of the boundary carefully. If $z_* = e^{\frac{t}{3}}$ then the boundary of the 2-cycle is exactly at the point on the base where the $\mathbb{C}^*$ fibration degenerates. In this case the 2-cycle and the 1-cycle of the $\mathbb{C}^*$ fibration together define a closed 3-cycle [61, 62]. Since the 1-cycle of the $\mathbb{C}^*$ fibration is always present the essential geometry of the 3-cycle is captured by the 2-cycle with boundary at $z_* = e^{\frac{t}{3}}$. This gets translated to finding minimal 2 surfaces in the corresponding $K3$ geometry with boundary being a circle on a particular elliptic fiber.

The connection with F-theory is now rather clear. In fact this elliptic fibration defines an F-theory background studied before in the context of non-BPS stable states in F-theory [59] and compactification of 5D E_n field theories on a circle [60]. As usual in the F-theory description, we can assign (p, q) charges to the 7-branes which correspond to (p, q) degenerating cycle of T^2 (which can be defined by choosing paths to a base point, z_0). In this case the charges are as shown in Fig. 45 [59, 60].Note that the (p, q) charge of the three degenerating cycles can be cyclically transformed by the $SL(2, \mathbf{Z})$ matrix ST, $(ST)^3 = 1$ [60].

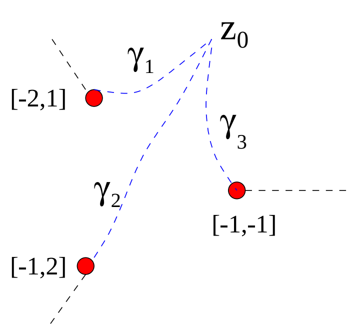

Figure 45:

The relation with the probe theory and its BPS states is now clear. The degenerating

[2]The existence of the 2nd type of closed 2-cycle is precisely the reason this fibration can be used to construct 5-brane web description of 5D E_0 field theory which can also be obtained via geometric engineering from the CY threefold $\mathcal{M}$ [63].

fibers at z_i define a 7-brane background and the degenerating 1-cycle at $z_* = e^{\frac{t}{3}}$ of the $\mathbb{C}^*$ fibration defines the position of the D3-brane. The 2-cycles with boundary are strings and string junctions stretched between the 7-branes and the D3-brane. BPS states in the D3-brane theory correspond to BPS string junctions which are the projections of holomorphic 2-cycles. Thus we see that D-branes wrapped on even dimensional cycles of $\mathcal{M}$ are mirror to states in the D3-brane worldvolume theory. This connection between sheaves on a non-compact CY and states in the field theory realized on a D3-brane in the presence of 7-branes was also studied in [62].

8.1.3 Soliton Numbers for $\mathbb{P}^2$ and its Blowups

As noted before, the elliptic fibration defined by eq. (8.12) is exactly the W-plane geometry of the massive LG theory mirror to sigma model with $\mathbb{P}^2$ target space. The vanishing cycles are just the cycles of the T^2 fiber, and so we can use the knowledge of the degeneration types to find the intersection number of vanishing cycles, and thus the soliton numbers of this theory. From Fig. 45 it follows that,

$$\gamma_i \circ \gamma_j = \begin{pmatrix} 0 & -3 & 3 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{pmatrix} . \tag{8.20}$$

Where we have written the intersection matrix as an upper triangular matrix. Similarly, as mentioned before, we can consider other non-compact CY threefolds $\mathcal{M}_n$ which are the total space of the canonical line bundle over the toric del Pezzo, $\mathcal{B}_n$. As for the case of $\mathbb{P}^2$, in this case as well the mirror is an elliptic fibration and a $\mathbb{C}^*$ fibration over the z-plane. The elliptic fibration is defined by the superpotential of the corresponding massive LG theory. The corresponding F-theory backgrounds and the D3-brane theory were studied in [64, 62]. Since the charges of the vanishing cycles are known for these cases we can compute the soliton counting matrix and compare with the matrices obtained from the collection of exceptional bundles. In the following we will denote the superpotential of the LG theory mirror to sigma model on X as W_X.

$\mathcal{B}_1$**:** There are four degenerate fibers in this case as shown in Fig. 46. Three out of four cycles are the same as before. The new degenerate fiber has charge $(-1,1)$. The intersection matrix is,

$$\gamma_i \circ \gamma_j = \begin{pmatrix} 0 & -1 & -3 & 3 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{8.21}$$

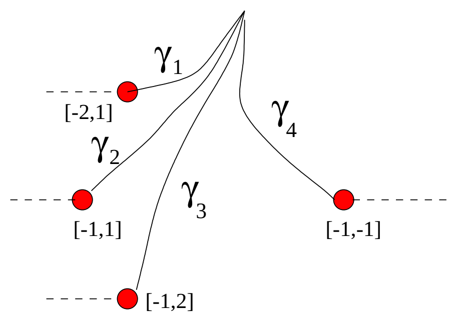

Figure 46: Positions of the degenerate fibers of $W_{\mathcal{B}_1}$

One can check that this matrix produces correct Ramond charges for the chiral fields.

$\mathcal{B}_2$**:** In this case there are five degenerate fibers as shown in Fig. 47. There are two mutually local (with the same charge) fibers. The intersection matrix is given by

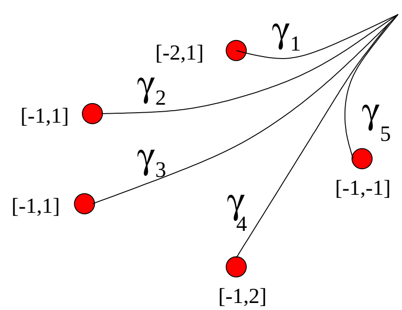

Figure 47: Positions of the degenerate fibers of $W_{\mathcal{B}_2}$.

$$\gamma_i \circ \gamma_j = \begin{pmatrix} 0 & -1 & -1 & -3 & 3 \\ 0 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{8.22}$$

$\mathcal{B}_3$**:** The six degenerate fibers in this case are shown in Fig. 48. In this case there are three mutually local fibers. The intersection matrix is given by

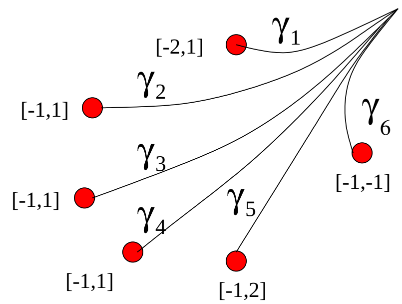

Figure 48: Positions of the degenerate fibers of $W_{\mathcal{B}_3}$.

$$\gamma_i \circ \gamma_j = \begin{pmatrix} 0 & -1 & -1 & -1 & -3 & 3 \\ 0 & 0 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} . \tag{8.23}$$

F_0**:** The four degenerate fibers in this case are shown in Fig. 49. The intersection matrix

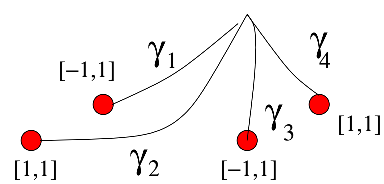

Figure 49: Positions of the degenerate fibers of W_{F_0}.

is given by

$$\gamma_i \circ \gamma_j = \begin{pmatrix} 0 & -2 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{8.24}$$

One can check that these matrices give the correct Ramond charge for the chiral fields.

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